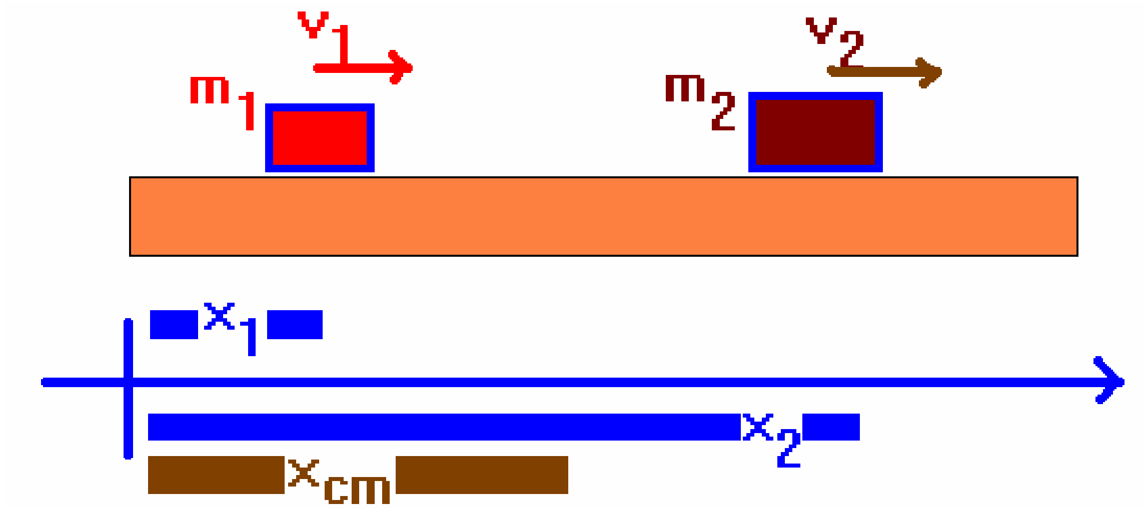
Notes for Conservation of translational momentum



Topic 1: Calculation of center of mass.

The center of mass is defined by

$$X_{cm} = \frac{\sum_{\text{all masses } (m_j) \text{ located at } x_j} m_j x_j}{\sum_{\text{all masses } (m_j)} m_j} = \frac{\sum_j m_j x_j}{\sum_j m_j}$$

This is nothing more than a weighted average. The concept is the same as if you have 5 A's and 4 B's … calculate your average grade.

According to Newton's laws, we have the following situation:

$\sum \vec{F}_{external} = M\vec{a}$ where M is the total mass of the system and a is the acceleration of the system. In fact, till now, we have actually been working with M located at the center of mass as defined above and a is the acceleration of this center of mass. Let's write Newton's law now in a different way:

$$\sum \vec{F} = M \frac{\Delta \vec{v}}{\Delta t}$$

$$\text{Calculus version: } \sum \vec{F} = M \frac{d\vec{v}}{dt}$$

The question here, however, is what exactly happens if no net external forces are acting on the system. From Newton's laws above, we have:

$$\vec{0} = M\frac{\Delta\vec{v}}{\Delta t} \Rightarrow M\Delta\vec{v} = \vec{0}$$

$$\text{Calculus version: } \vec{0} = M\frac{d\vec{v}}{dt}$$

Looking at the expressions above, it is very interesting to observe that in the absence of net external forces, we have something conserved, namely the product of mass and velocity. This, for us, will define a new quantity in physics, namely the momentum. This is given by

$$\vec{P} \equiv M\vec{v}$$

And, in the absence of external forces we thus conserve momentum, or:

$$\Delta\vec{P} = \vec{0}.$$

Ok, what does this have to do with the center of mass that we began this discussion with? Let's see:

$$X_{cm}\sum_j m_j = \sum_j m_j x_j \Rightarrow MX_{cm} = \sum_j m_j x_j$$

where M is the total mass of the system. Let's see how this changes in time:

$$M\frac{\Delta\vec{X}_{cm}}{\Delta t} = \sum_j m_i \frac{\Delta\vec{x}_j}{\Delta t}$$

$$\text{Calculus Version: } M\frac{d\vec{X}_{cm}}{dt} = \sum_i m_i \frac{d\vec{x}_i}{dt}$$

You can recognize the velocity now pretty easily so this becomes:

$$M\vec{v}_{cm} = \sum_j m\vec{v}_j$$

Let's write this, however, in terms of the momentum which we have defined above:

$$\vec{P}_{cm} = \sum_j \vec{P}_j$$

What this says is that the motion of the system can be reduced to a consideration of the motion of the center-of-mass of the system. This is pretty nice since it means that we don't always need to follow the details of every single mass of a system … we might be able to just follow what the center of mass of the system does.

Ok, let's fit this together with the conservation of momentum that we talked about earlier. If no external forces are present on the system, the total translational momentum of the center-of-mass of the system will remain unchanged. Thus,

$$\text{if } \vec{F}_{external} = 0 \text{ then } \Delta\vec{P}_{cm} = \vec{0}.$$

$$\text{If now } \Delta\vec{P}_{cm} = \sum_j \Delta\vec{P}_j = \vec{0} \Rightarrow \sum_j \Delta\vec{P}_j = \vec{0}.$$

This is a very useful result since we can now also look at the details of the internal dynamics of a system. It is also useful to note that if external forces are present, we can look at the motion of the center of mass of the system in order to describe how the system moves through time.

Let's look at how to use this result:

$$\Delta\vec{P} = \vec{0} \Rightarrow \sum_j \vec{P}_{j,final} = \sum_j \vec{P}_{j,initial}$$

1-dimensional case:

The mass of the system is not always constant. The nice thing about looking at momentum is that when Newton's law is cast in terms of momentum, it can deal with changing mass, or:

$$\sum \vec{F} = \tfrac{\Delta\vec{P}}{\Delta t}$$

Calculus version: $\sum \vec{F} = \tfrac{d\vec{P}}{dt}$

For now, however, we're going to concentrate on the case that the motion is 1 dimensional and no external forces are present.

In one dimension, with no external forces present, we then have:

$$\Delta P = 0 \Rightarrow P_{before} = P_{after}$$

The following discussion now assumes **No External Forces**.

This will enable us to describe collisions in one dimension. There will be two types of collisions which we consider, namely totally elastic or totally inelastic. **No matter what type of collision is involved, the total momentum will be conserved.**

The big distinction between these two collision types is this:

(1) for totally elastic collisions, kinetic energy is also conserved (two steel balls hit each other and bounce off … sparks don't fly, no sound is heard, no heat is generated, things don't start rotating, etc).

(2) for totally inelastic collisions, kinetic energy will not be conserved (two balls hit each other and stick together).

The easiest of these two collisions to discuss is the totally inelastic collision. Let's do this now.

Two balls (m_1, v_1) and (m_2,v_2) collide and stick together after the collision. How fast does the system move after the collision?

Since no external forces are present, $\vec{P}_{before} = \vec{P}_{after}$. This then gives us the following situation:

$$m_1 v_{1,b} + m_2 v_{2,b} = (m_1 + m_2) v_{after}$$

This is, of course, easily solved for the velocity after the collision:

$$v_{after} = \frac{m_1 v_{1,b} + m_2 v_{2,b}}{m_1 + m_2}$$

On the web page, I have provided an animation of this type of collision for one specific case, namely where $m_1=m_2$. This is easy enough to do right here also:

If $m_1=m_2$, then

$$v_{after} = \frac{m_1 v_{1,b} + m_1 v_{2,b}}{m_1 + m_1} = \tfrac{1}{2}\left(v_{1,b} + v_{2,b}\right)$$

In particular, let's consider the case where the second mass is not moving before the collision. Then:

$$v_{after} = \tfrac{1}{2} v_{1,before}$$

It's pretty easy to calculate how much kinetic energy was converted into other forms of energy in this example:

$$K_{before} = \tfrac{1}{2} m v_b^2 \text{ and } K_{after} = \tfrac{1}{2}(2m)(\tfrac{1}{2} v_b)^2 = \tfrac{1}{4} m v_b^2 = \tfrac{1}{2}\left[\tfrac{1}{2} m v_b^2\right] = \tfrac{1}{2} K_{before}$$

From this, we can see:

$$\Delta K = K_{after} - K_{before} = K_{before}\left(\tfrac{1}{2} - 1\right) = -\tfrac{1}{2} K_{before}$$

We thus see that the fractional loss of Kinetic Energy in this case is:

$$\tfrac{\Delta K}{K} = -\tfrac{1}{2} \Rightarrow 50\% \text{ loss}$$

Totally elastic collisions

In totally elastic collisions, we have that kinetic energy and momentum are conserved. Thus,

$\Delta\vec{P} = \vec{0}$ and $\Delta K = 0$.

It is useful to see how kinetic energy can be re-expressed in terms of momentum. The result (you can verify this) is:

$$K = \frac{\vec{P}\bullet\vec{P}}{2m}$$

Let's apply this now to a more specific problem in one dimension where $K = \frac{P^2}{2m}$

Suppose a mass collides elastically with a second mass. How does each of the masses move after the collision?

(notation: i means "initial, f means "final")

$$\text{Before: } P_1 = m_1 v_{1,i} \;\; P_2 = m_2 v_{2,i} \;\; K_1 = \tfrac{1}{2} m_1 v_{1,i}^2 \;\; K_2 = \tfrac{1}{2} m_2 v_{2,i}^2$$

$$\text{After: } P_1 = m_1 v_{1,f} \;\; P_2 = m_2 v_{2,f} \;\; K_1 = \tfrac{1}{2} m_1 v_{1,f}^2 \;\; K_2 = \tfrac{1}{2} m_2 v_{2,f}^2$$

You'll immediately see that we have 4 velocities here, but only 2 equations. This means that for any correctly stated problem, we're going to have to know something about at least two of these velocities in order to solve for the other two.

Here's something we can always do, however! We can always transform to a coordinate frame in which this is true:

$$v_{2,i} = 0$$

How to do this? Simply walk along with the second mass before the collision. Notice not all observers measure the same kinetic energy!

Note, however, that you can always require conservation of momentum, even if you don't know the details of the type of collision. I've provided two accident problems to show this.

With this restriction, our equations reduce to the following:

$$\Delta P = 0 \Rightarrow m_1 v_{1,i} = m_1 v_{1,f} + m_2 v_{2,f}$$

and

$$\Delta K = 0 \Rightarrow \tfrac{1}{2} m_1 v_{1,i}^2 = \tfrac{1}{2} m_1 v_{1,f}^2 + \tfrac{1}{2} m_2 v_{2,f}^2$$

These equations can be inverted to provide the final velocities in terms of the initial velocity:
The steps showing this are reproduced as a note at the end of these notes.
The results are, however:

$$v_{1,f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1,i} \text{ and } v_{2,f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1,i} \, .$$

The expressions are a little bit more complicated if you do not have one of the masses initially at rest … but as I said, you can always arrange this to be so.
Applications of these two equations are shown in the worksheet.

Problems in two dimensions

In two dimensions, we still have conservation of P and K for totally elastic collisions. However, the introduction of additional variables may force many details to be known before the collision. There is, however, one 2-dimensional situation which is relatively straight forward, namely a glancing totally elastic collision between two identical masses with mass 2 initially at rest. Let's see that this produces:

P conservation gives:

$$\vec{P}_{before} = \vec{P}_{after} \Rightarrow m_1 \vec{v}_{1,i} = m_1 \vec{v}_{1,f} + m_2 \vec{v}_{2,f}$$

K conservation gives:

$$K_{before} = K_{after} \Rightarrow \tfrac{1}{2} m_1 \vec{v}_{1,i} \bullet \vec{v}_{1,i} = \tfrac{1}{2} m_1 \vec{v}_{1,f} \bullet \vec{v}_{1,f} + \tfrac{1}{2} m_2 \vec{v}_{2,f} \bullet \vec{v}_{2,f}$$

Since the masses are all identical, these two equations reduce to:

$$\vec{v}_{1,i} = \vec{v}_{1,f} + \vec{v}_{2,f}$$

$$\vec{v}_{1,i} \bullet \vec{v}_{1,i} = \vec{v}_{1,f} \bullet \vec{v}_{1,f} + \vec{v}_{2,f} \bullet \vec{v}_{2,f}$$

Take the "dot" product of the first equation with itself:

$$\vec{v}_{1,i} \bullet \vec{v}_{1,i} = (\vec{v}_{1,f} + \vec{v}_{2,f}) \bullet (\vec{v}_{1,f} + \vec{v}_{2,f}) = \vec{v}_{1,f} \bullet \vec{v}_{1,f} + \vec{v}_{2,f} \bullet \vec{v}_{2,f} + 2\vec{v}_{1,f} \bullet \vec{v}_{2,f}$$

Compare this to the kinetic energy equation and you can see that the only way to both conserve momentum and kinetic energy is to require:

$$\vec{v}_{1,f} \bullet \vec{v}_{2,f} = 0 \Rightarrow \cos(\theta) = 0 \Rightarrow \theta = 90^0 \, .$$

Thus the angle between the reactants must be 90^0 in this case. We can't, however, really say too much about the final velocities unless at least one of the final velocities is measured.

Algebraic Details:

$$\Delta P = 0 \Rightarrow m_1 v_{1,i} = m_1 v_{1,f} + m_2 v_{2,f} \; (1)$$

and

$$\Delta K = 0 \Rightarrow \tfrac{1}{2} m_1 v_{1,i}^2 = \tfrac{1}{2} m_1 v_{1,f}^2 + \tfrac{1}{2} m_2 v_{2,f}^2 \; (2)$$

Solve equation 1 for v_{1f}:

$$v_{1f} = v_{1i} - \tfrac{m_2}{m_1} v_{2f}$$

Put this into (2):

$$m_1 v_{1,i}^2 = m_1 \left[v_{1i} - \tfrac{m_2}{m_1} v_{2f} \right]^2 + m_2 v_{2,f}^2$$

$$\Rightarrow v_{1,i}^2 - v_{1i}^2 + 2\tfrac{m_2}{m_1} v_{1i} v_{2f} - \tfrac{m_2^2}{m_1^2} v_{2f}^2 = \tfrac{m_2}{m_1} v_{2f}^2$$

$$\Rightarrow 2\tfrac{m_2}{m_1} v_{1i} v_{2f} = \tfrac{m_2}{m_1} v_{2f}^2 + \tfrac{m_2^2}{m_1^2} v_{2f}^2$$

$$\Rightarrow 2\tfrac{m_2}{m_1} v_{1i} = v_{2f} \left[\tfrac{m_2}{m_1} + \tfrac{m_2^2}{m_1^2} \right]$$

$$\Rightarrow v_{2f} = v_{1i} \frac{2\frac{m_2}{m_1}}{\left[\frac{m_2}{m_1} + \frac{m_2^2}{m_1^2}\right]} = v_{1i} \frac{2}{\frac{m_1}{m_2}\left[\frac{m_2}{m_1} + \frac{m_2^2}{m_1^2}\right]} = v_{1i} \frac{2}{\left[1 + \frac{m_2}{m_1}\right]} = v_{1i} \frac{2m_1}{[m_1 + m_2]}$$

We can thus solve for v_{1f}:

$$v_{1f} = v_{1i} - \tfrac{m_2}{m_1} v_{1i} \frac{2m_1}{[m_1 + m_2]} \Rightarrow v_{1f} = v_{1i} \left[\frac{m_1 + m_2 - 2m_2}{m_1 + m_2} \right] = v_{1i} \left[\frac{m_1 - m_2}{m_1 + m_2} \right]$$