Special Calculus notes related to centripetal acceleration

A point moving in uniform circular motion is described by a position vector given by:

$$\vec{R} = b\cos(\omega t)\hat{i} + b\sin(\omega t)\hat{j}$$

The velocity at each instant in time is given by the derivative of this:

$$\vec{v} = \frac{d\vec{R}}{dt} = -\omega b\sin(\omega t)\hat{i} + \omega b\cos(\omega t)\hat{j}$$

The velocity here is tangent to the curve and it is the velocity that determines what the curve ultimately looks like. The vector pointing in the direction of the velocity vector is given by the unit vector pointing in the direction of the velocity (this is called the tangent vector):

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{-\omega b\sin(\omega t)\hat{i} + \omega b\cos(\omega t)\hat{j}}{\sqrt{\vec{v}\bullet\vec{v}}} = \frac{-\omega b\sin(\omega t)\hat{i} + \omega b\cos(\omega t)\hat{j}}{\omega b} \Rightarrow \hat{T} = -\sin(\omega t)\hat{i} + \cos(\omega t)\hat{j}$$

Now you would like to say that this vector points in the angular direction, and you indeed can say that simply by defining the unit vector in the angular direction:

$$\hat{\theta} = -\sin(\omega t)\hat{i} + \cos(\omega t)\hat{j}$$

You would also like to know that the position vector in the radial direction is then given by:

$$\hat{R} = \frac{\vec{R}}{|\vec{R}|} = \cos(\omega t)\hat{i} + \sin(\omega t)\hat{j}$$

You can also easily prove that these two vectors are perpendicular to each other:

$$\hat{R}\bullet\hat{\theta} = -\sin(\omega t)\cos(\omega t) + \cos(\omega t)\sin(\omega t) = 0$$

An additional detail: you might want to replace the argument by theta if you're not dealing with problems involving motion.

Now when the point is moving around a circle, the velocity is given by:

$$\vec{v} = |\vec{v}|\hat{T} = |\vec{v}|\hat{\theta}$$

where the last equality results only in this case for a particle moving on a circular path. We can now find the acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d|\vec{v}|}{dt}\hat{\theta} + |\vec{v}|\frac{d\hat{\theta}}{dt}$$

If the speed of the particle is constant, we then have:

$$\vec{a} = |\vec{v}|\left[-\omega\cos(\omega t)\hat{i} - \omega\sin(\omega t)\hat{j}\right] = -\omega|\vec{v}|\hat{R}$$

In calculus III, this would be referred to as the normal component of the acceleration, in physics we call this the centripetal acceleration.

Thus: if v is the speed of the particle, we then have:

$$|\vec{a}| = \omega v$$

Now for a particle undergoing circular motion, it will have a connection between the distance it goes through and its velocity. The distance in one revolution is given by:

$$s = \int ds = \int_{\theta=0}^{2\pi} b\,d\theta = 2\pi b$$

And so the time for one revolution is given by:

$$vT = 2\pi b \Rightarrow v = \frac{2\pi b}{T} = \omega b$$

Thus the centripetal acceleration is given by:

$$a_c = \omega v = \omega^2 b$$

If your circle has a radius R instead of b, we then have:

$$a_c = \omega^2 R$$