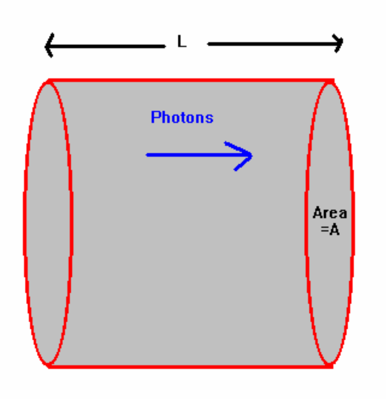
For a photon, p=U/c

Revised Fall 2006

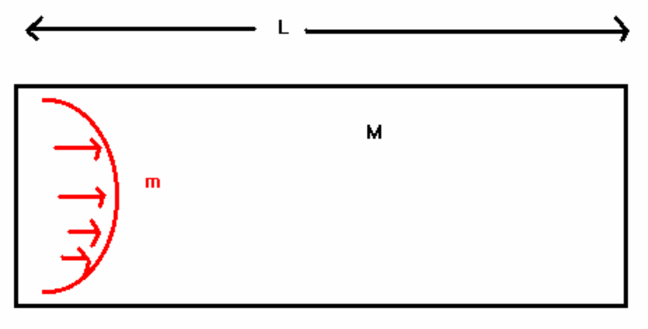


Consider the cylinder shown to the left. Imagine that one side of the cylinder emits a massless cylinder of photons, and the momentum from this massless cylinder of photons is completely absorbed by the other end of the cylinder. The photons carry away momentum and impart this momentum to the cylinder. The work done is given by the total energy U of the photon gas. By the work-energy theorem, we then have: $\vec{F} \bullet \vec{x} = \Delta K = U$.

The length containing this photon gas is L as shown in the diagram. So, $FL = U$. According to Newton's laws, a force and a change in momentum are related by $\vec{F} = \frac{d\vec{P}}{dt}$ where P is the momentum associated with the cylinder of photon gas and t is time. We use this in the second equation to provide: $\frac{d\vec{P}}{dt} L = U$. Since the photons are completely absorbed, in a time t, we then have $\frac{dP}{dt} \approx \frac{\Delta P}{\Delta t} = -\frac{P}{\Delta t}$ where we've now allowed a sufficient Δt for all the photons to be absorbed. Thus, the connection with the total energy becomes $-P\frac{L}{\Delta t} = U$ where U is now the energy loss of the photon gas. We can eliminate the - sign if we refer to the initial energy of the photon gas. The quantity $L/\Delta t$ is the speed that the photon gas moves with, namely the speed of light (c). This then gives us the initial energy of the photon gas as $Pc = U$, or written in it more usual form, we have the connection between a photon and the energy of that photon as $p = \frac{U}{c}$. You'll often see this written as $p = \frac{E}{c}$ where E and U have the same meaning. At times, however, it is important to keep in mind that U might refer to more than one photon while E refers to a single photon. The connection is U=nE with E the energy of a single photon.

For mass at rest, $E=mc^2$.

This argument can be found in "Contemporary College Physics," third edition by Jones/Childers page 801.



This is a thought experiment that was originally due to Einstein. Consider a closed box of length L and mass M. We are in a frame of reference in which the box is initially not moving and friction is not present in this situation, so that the box is completely free to move if it needs to. Suppose one end of the box emits a bunch of photons instantaneously. (this is fine since if the box is above 0K, it will be continually radiating photons to the environment). We have just shown above that the photons will carry away momentum so that the box must recoil, and the amount of the recoil is given by the conservation of momentum: $\Delta\vec{P}_{\text{system}} = 0 \Rightarrow P_{\text{box}} = P_{\text{photon}}$. Thus, $P_{\text{box}} = Mv = P_{\text{photon}} = \frac{E}{c}$. v is the recoil velocity of the box and E is the energy carried away by the photons. Next, the photons are completely absorbed by the other end of the box within a time given approximately by $t \approx \frac{L}{c}$. During this short time, the box moved through a distance x which is given by $x = vt = \frac{P_{photon}}{M}\frac{L}{c}$. This distance x is also related to an "imagined mass" which Einstein associated the photon: mL=Mx where m is the "imagined photon mass". Without this imagined mass, it is clear that the closed system could become unbalanced or the center of mass of the box-photon might suddenly start moving. This mass m actually represents a decrease in the total mass M of the box which is converted into energy We can relate the momentum of the photon to its energy so the distance the box moved is related to the energy by $x = \frac{E}{M}\frac{L}{c^2}$.. We can now use the x which we obtained first to find: $mL = M\frac{E}{M}\frac{L}{c^2}$. This then simplifies to give the famous result: $E = mc^2$. What this represents is the answer to the following question: suppose the side of the box totally radiated. How much energy would be lost by the conversion of this matter into photons.

Relativistic momentum for substantive material

I have provided notes related to relativistic momentum. The result is that the classical definition for momentum is modified to become:

$$\vec{p} = \frac{m\vec{u}_0}{\sqrt{1-\left(\frac{u_0}{c}\right)^2}}$$

where the mass is regarded as a quantity which all frames would measure to be the same value and u_0 is the speed of the particle as measured by an observer in the frame of reference in which the momentum is being measured.

Find the relativistic expression for the kinetic energy of a particle.

From Newton's law, we define a force in terms of the rate of change of momentum that this force results in:

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(\gamma_u m\vec{u})$$

We find the kinetic energy from the work energy theorem which says:

$$\Delta K = \text{Work} = \int_i^f \vec{F} \bullet d\vec{s}$$

where s is a path and F is conservative (here, by assumption). Substituting for F (and starting with zero velocity and zero kinetic energy), we have:

$$K = \int_i^f \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\right) \bullet d\vec{s}\,.$$

Now, the differential path is best related to a translation here since we strictly want to avoid accelerations. This is ok also since our force is assumed to be conservative. This allows the expression to be written as:

$$K = \int_i^f m\frac{d}{dt}\left(\frac{u}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\right)ds = m\int_i^f \frac{d}{dt}\left(\frac{u}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\right)ds$$

We also know what the path is: it is the path of motion of the particle. Thus:

$$ds = d(ut)$$

The easiest way to avoid many problems here is to agree with your author that:

$$ds = udt$$

Perhaps in the future, I'll explore this more. For now, we will accept this.

The integral for the kinetic energy then becomes:

$$K = \int_i^f m\frac{d}{dt}\left(\frac{u}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\right)ds = m\int_i^f u\frac{d}{dt}\left(\frac{u}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\right)dt = m\int_0^{\gamma_u u} ud\left[\gamma_u u\right] = m\int_0^u \gamma_u u\,du + m\int_1^{\gamma_u} u^2 d\gamma_u$$

I evaluated the first integral with the integrator:

$$\int \gamma_u u\,d\left[u\right] = \int \frac{u}{\sqrt{1-\frac{u^2}{c^2}}}du = -c^2\sqrt{1-\frac{u^2}{c^2}}\Bigg|_0^u = -c^2\sqrt{1-\frac{u^2}{c^2}} + c^2 = c^2\left[1-\frac{1}{\gamma_u}\right]$$

We can write u in terms of γ_u:

$$\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \Rightarrow \gamma_u^2 = \frac{1}{1-\frac{u^2}{c^2}} \Rightarrow 1-\frac{u^2}{c^2} = \frac{1}{\gamma_u^2} \Rightarrow 1-\frac{1}{\gamma_u^2} = \frac{u^2}{c^2} \Rightarrow u^2 = c^2\left[1-\frac{1}{\gamma_u^2}\right]$$

This lets us evaluate the second integral:

$$\int u^2 d\gamma_u = c^2\int\left[1-\frac{1}{\gamma_u^2}\right]d\gamma_u = c^2\left\{\gamma_u + \frac{1}{\gamma_u}\right\}\Bigg|_1^{\gamma_u} = c^2\left[\gamma_u + \frac{1}{\gamma_u} - 1 - 1\right] = c^2\left[\gamma_u + \frac{1}{\gamma_u} - 2\right]$$

If we add the results, we have:

$$K = mc^2\left[1-\frac{1}{\gamma_u}+\gamma_u+\frac{1}{\gamma_u}-2\right] = mc^2\left[\gamma_u - 1\right]$$

Notice that, in contrast to your author's statement, integration by parts was not necessary.

It is not necessary to use the integrator for the first integral. Here is how to do it:

$$\text{Let } I_1 \equiv \int_0^w \gamma w dw. \text{ Then, } I_1 = \int_0^w \frac{wdw}{\sqrt{1-\beta^2}} = c^2\int_0^w \frac{\beta d\beta}{\sqrt{1-\beta^2}}.$$

The following came directly from Ed Mosley (Thanks! We miss you!)

This is evaluated with a trigonometric substitution:

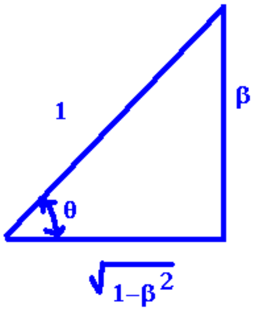
let $\sin(\theta)=\beta$. Then $I_1 = c^2\int\frac{\sin(\theta)\cos(\theta)d\theta}{\sqrt{1-\sin^2(\theta)}} = c^2\int\sin(\theta)d\theta = -c^2\cos(\theta)$.

Now do the reverse substitution: Consider the triangle shown to the right. We have then that $\cos(\theta) = \sqrt{1-\beta^2}$.

So $I_1 = -c^2\sqrt{1-\beta^2}\Big|_0^w = -c^2(\sqrt{1-\beta^2}-1) = c^2(1-\frac{1}{\gamma})$.

Thus, $I = \gamma w^2 - c^2(1-\frac{1}{\gamma}) = c^2\gamma(\frac{w^2}{c^2}-\frac{1}{\gamma}+\frac{1}{\gamma^2})$

$I = c^2\gamma(\frac{w^2}{c^2}-\frac{1}{\gamma}+\frac{1}{\gamma^2}) = \gamma c^2(\beta^2-\frac{1}{\gamma}+1-\beta^2) = c^2(\gamma-1)$



So we can now find the relativistic kinetic energy to be $K = mc^2(\gamma_u - 1)$. Let's look at one of these terms, namely the mc^2 term. This is subtracted away from the kinetic energy and corresponds to the rest mass energy, $E_0=mc^2$. Thus, we have $K + E_o = \gamma_u mc^2 = E$ where E is the total energy of the particle (moving without any external potentials). This E is exactly the same energy that was obtained above.

Finding an equivalent expression for PC=E for substantive material.

The goal here is to verify $E^2=p^2c^2+m^2c^4$.

We have the relativistic momentum given by $p=\gamma_u mu$ and the relativistic kinetic energy given by $K=mc^2(\gamma_u-1)$ which provided us with the total energy $K+E_o=\gamma_u mc^2=E$ of a particle of mass m. Re-writing the last expression: $E^2=K^2+2KE_o+E_o^2=\gamma_u^2m^2c^4$. Look at P^2: $p^2c^2=\gamma_u^2m^2u^2c^2$. So ...

$$p^2c^2+m^2c^4=\gamma_u^2m^2u^2c^2+m^2c^4=\gamma_u^2m^2c^4(\tfrac{u^2}{c^2}+\tfrac{1}{\gamma_u^2})=\gamma_u^2m^2c^4(\tfrac{u^2}{c^2}+1-\tfrac{u^2}{c^2})=\gamma_u^2m^2c^4=E^2.$$

This thus shows the second of the energy relations, namely $E^2=p^2c^2+m^2c^4$ for material which has mass and E=pc for material without mass such as photons.