Relativistic Momentum

Frank is in a fixed frame with a ball of mass m while Mary is in a moving frame (moving at a speed v along the +x axis as seen by Frank) also with a ball of mass m.

A classical description of the problem

(There are animations of this on the web site)

Frank throws his ball up along the y axis with a velocity $+u_0$ while Mary throws her ball down along the y axis with a velocity $-u_0$. The balls collide exactly correctly (head on) elastically and bounce off each other. After the collision, Mary measure her ball to have a velocity $+u_0$ while Frank measure his ball to have a velocity of $-u_0$. Let's calculate the change in momentum as seen by different frames.

According to Frank, the change in momentum for his ball is:

$$\Delta p_{F,b_F} = p_a - p_b = -mu_0 - mu_0 = -2mu_0$$

According to Frank, the change in momentum for Mary's ball is:

$$\Delta P_{F,b_M} = p_a - p_b = +mu_0 - (-mu_0) = +2mu_0$$

Classically, according to Frank, the total change in momentum is zero and momentum is conserved.

Likewise, let's see what Mary observes.

According to Mary, the change in momentum for her ball is:

$$\Delta p_{M.b_M} = p_a - p_b = +2mu_0$$

According to Mary, the change in momentum for Frank's ball is:

$$\Delta p_{M,b_F} = p_a - p_b = -2mu_0$$

Classically, according to Mary, the total change in momentum is also zero and momentum is conserved.

However, we know that according to special relativity, after Mary throws her ball, there is a relativistic correction to the velocity addition that needs to be taken into account. I want to do this now to show clearly the problem.

Mary's frame is moving with a velocity v parallel to the x-axis. Upon throwing her ball, she observes a velocity vector given by:

$$\vec{u}_{M,b_M} = v\hat{x} - u_0\hat{y}$$

But what is the velocity observed by Frank?

This comes from the Lorentz transformation:

$$x = \gamma(x' + vt') : y = y' : z = z' : t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\beta^2}}$$

$$dx = \gamma(dx' + v\,dt') : dy = dy' : dz = dz' : dt = \frac{dt' + \frac{v}{c^2}dx'}{\sqrt{1-\beta^2}}$$

the velocity transformation for motion along the y-direction is what is important here, however. This is given by:

$$u_0 = \frac{dy}{dt} = \frac{dy'}{\frac{dt' + \frac{v}{c^2}dx'}{\sqrt{1-\beta^2}}} = \sqrt{1-\beta^2}\,\frac{\left(\frac{dy'}{dt'}\right)}{1+0} = u_0'\sqrt{1-\beta^2}$$

Ok, this means that Frank measures the y-component of the velocity of Mary's ball just before the collision to be:

$$U_{0_{F,b_M}} = -u_0'\sqrt{1-\beta^2}$$

Frank measures the y-component of the velocity of Mary's ball just after the collision to be:

$$u_{0_{F,b_M}} = +u_0'\sqrt{1-\beta^2}$$

According to Frank, the change in momentum of Mary's ball would then be given by:

$$\Delta p_{F,b_M} = p_a - p_b = 2mu_0'\sqrt{1-\beta^2}$$

Of course, Mary is going to be measuring a velocity u_0 in the y-direction. This gives:

$$\Delta p_{F,b_M} = 2mu_0\sqrt{1-\beta^2}$$

Frank, on the other hand, is going to measure the same velocity as he did before and thus the change in momentum as measured by Frank's ball will be given by:

$$\Delta p_{F,b_F} = -2mu_0$$

We require the total change in momentum to be zero which then gives:

$$\Delta p_{total} = \Delta p_{F,b_F} + \Delta p_{F,b_M} = -2mu_0 + 2mu_0\sqrt{1-\beta^2} = 2mu_0\left[\sqrt{1-\beta^2} - 1\right]$$

which does equal zero if β is zero. Otherwise, a real problem exists.

I don't like the way your author pointed out this difficulty: there is no difficulty classically. It is only when the relativistic velocity addition is used that the problem is apparent. (see page 61).

Since we like to believe special relativity is correct and we also like to believe momentum is conserved, the only conclusion that can be drawn from this demonstration is that the simple form we used for momentum must not be the entire truth.

Assume a relativistic version of the momentum may appear as:

$$\vec{p} = \Gamma(v, u_0)m\vec{u}_0$$

The question that needs to be answered then is what is the form for the relativistic factor that would conserve momentum. Rather than derive this, I will say that the correction is:

$$\Gamma(u_0) = \frac{1}{\sqrt{1-\left(\frac{u_0}{c}\right)^2}} \equiv \gamma_u$$

Your author does not use the last symbol but I have here since you may see this in other text books. It is important to note that this is not the "gamma" factor that we have worked with earlier, since the velocity is different: it does not involve the velocity of the moving frame at all.

Now I want to show you that this form of the momentum:

$$\vec{p} = \frac{m\vec{u}_0}{\sqrt{1-\left(\frac{u_0}{c}\right)^2}}$$

does conserve momentum in the previous example of the Frank and Mary collision.

If we accept this for the momentum (y component), the momentum observed by Frank would be:

(a) for Frank's ball:

$$p_{y:F,b_F} = \frac{mu_0}{\sqrt{1-\left(\frac{u_0}{c}\right)^2}}$$

so the total change in momentum that Frank observes for Frank's ball is then:

$$\Delta p_{y:F,b_F} = -2\frac{mu_0}{\sqrt{1-\left(\frac{u_0}{c}\right)^2}}$$

(b) for Mary's ball:

In this case, we still have to apply the relativistic velocity addition result:

Frank observes a speed for Mary's ball which is given by:

$$u_{0_{F,b_M}} = u_0\sqrt{1-\beta^2}$$

But there is an additional complication here: we need to find the speed of Mary's ball, including the x-component, as reported by Frank based upon Mary's measurements and Frank's observation of the motion of Mary's frame. Whew!

Watch this step very closely!

Here is the answer for this velocity:

$$\vec{u}_0 = v\hat{x} + u_0\sqrt{1-\beta^2}\,\hat{y} \Rightarrow u_0 = \sqrt{v^2+\left(1-\beta^2\right)u_0^2}$$

The momentum (y component) of Mary's ball would then be given by:

$$p_{y:F,b_M} = \frac{mu_0\sqrt{1-\beta^2}}{\sqrt{1-\left(\frac{u_0'}{c}\right)^2}} = \frac{mu_0\sqrt{1-\beta^2}}{\sqrt{1-\left(\frac{\sqrt{v^2+\left(1-\beta^2\right)u_0^2}}{c}\right)^2}}$$

I can, and should, simplify this a bit:

$$p_{y:F,b_M} = \frac{mu_0\sqrt{1-\beta^2}}{\sqrt{1-\frac{v^2+\left(1-\beta^2\right)u_0^2}{c^2}}} = \frac{mu_0\sqrt{1-\beta^2}}{\sqrt{1-\beta^2-\frac{\left(1-\beta^2\right)u_0^2}{c^2}}} = \frac{mu_0\sqrt{1-\beta^2}}{\sqrt{1-\beta^2-\frac{u_0^2}{c^2}+\beta^2\frac{u_0^2}{c^2}}} = \frac{mu_0\sqrt{1-\beta^2}}{\sqrt{\left(1-\beta^2\right)\left(1-\frac{u_0^2}{c^2}\right)}} = \frac{mu_0}{\sqrt{\left(1-\frac{u_0^2}{c^2}\right)}}$$

(If you follow this as you should for Mary's point of view, then you will appreciate the power of Mathtype).

(look at your assignment: problem 62 on page 82).

Now let me show that the total change in this momentum is zero.

$$\Delta p_y = \Delta p_{y:F,b_F} + \Delta p_{y:F,b_M} = -2\frac{mu_0}{\sqrt{1-\left(\frac{u_0}{c}\right)^2}} + 2\frac{mu_0}{\sqrt{\left(1-\frac{u_0^2}{c^2}\right)}} = 0$$

The conclusion of this is then that this form for the relativistic momentum:

$$\vec{p} = \frac{m\vec{u}_0}{\sqrt{1-\left(\frac{u_0}{c}\right)^2}}$$

seems to conserve momentum in the simple case considered and may well work in all of them. I can always transform to a frame like Frank's and look at this type of collision so I would expect that there can be little doubt concerning the generality of this result. You must, however, be quite careful about how you include the relativistic velocity additions, as you have seen.

I am also in agreement with your author to leave mass as an invariant. We will not speak about relativistic mass in our course, but we will speak about the invariant mass moving at high speeds.