Important Notes regarding 1-dimensional motion.

Notations: (you will need to understand these for the entire course)

$Q_f \equiv$ "final value of Q" : $Q_i \equiv Q_0 \equiv$ "initial value of Q" : $\Delta Q \equiv Q_f - Q_i$: $Q_x =$ "value of Q along the x direction"

$\langle Q \rangle$ denotes the time average value of the variable Q.

$Q \equiv$ means that the quantity Q is defined as what follows.

$\vec{Q}$ denotes a **vector** quantity while $\hat{q}$ denotes a unit vector.

Q_x is the portion of the vector pointing along the x direction. (called the "x component")

These equations assume a constant acceleration and the initial time is $t_i=0$.

$$x(t) = x_0 + v_0 t + \tfrac{1}{2} a_x t^2$$

$$v_x(t) = v_{0,x} + a_x t$$

$$v_x^2 = v_{0,x}^2 + 2a_x(\Delta x)$$

Average Velocity

$$V_{avg,x} \equiv \langle V \rangle \equiv \tfrac{\Delta x}{\Delta t}$$

The last equation is **always** true. In fact, if you know the average velocity, then:

$$\Delta x = \langle v \rangle \Delta t$$

In my notation, the <v> means the average of the variable.

There are two somewhat important results related to the average velocity in systems with a constant acceleration. These are:

$$\langle v \rangle = v_0 + \tfrac{1}{2} at \text{ and } \langle \Delta v \rangle = \tfrac{1}{2}\{v_f - v_0\}.$$

There are a number of ways to define the average velocity. Ultimately, I suggest you use the following. The **average velocity** is given by the total **displacement** divided by the **time** that it takes for this displacement to occur. Remember: displacement is a vector while distance is a scalar. The **average speed** would be given by the total **distance** divided by the **time** that it takes for the object to move this distance. When you have a system characterized by ***a constant acceleration***, the average velocity is given by

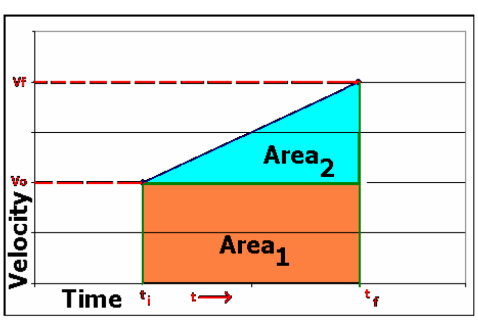
$$\langle v \rangle = v_0 + \tfrac{1}{2} at$$

You can then see that the distance traveled with a constant acceleration is given by:

$$\Delta x = \langle v \rangle t = v_0 t + \tfrac{1}{2} at^2 \Rightarrow x_f = x_0 + v_0 t + \tfrac{1}{2} at^2$$

Again, this is only valid for the special case of a constant acceleration. Another result is for the case of a constant acceleration, the average change in velocity is given by:

$$\langle \Delta v \rangle = \tfrac{1}{2}\{v_f - v_0\}.$$



Non-calculus proof:

The "why" behind these is the following: if you make a plot of the velocity vs. position for a constant acceleration, you would obtain something that may look like the graph shown. The area under this curve represents the total distance. The "orange" part has an area given by $\text{Area}_1 = v_0 \Delta t$. The "blue" part has an area given by $\text{Area}_2 = \tfrac{1}{2}[\Delta t][\Delta v]$. Thus the total distance traveled in this time interval is given by:

$$\Delta x = \text{Area}_1 + \text{Area}_2 = v_0\Delta t + \tfrac{1}{2}[\Delta t][\Delta v]$$

The change in velocity is then (for a constant acceleration) given by $\Delta v = a\,\Delta t$. We thus have that the change in position is given by:

$$\Delta x = v_0\Delta t + \tfrac{1}{2}a[\Delta t]^2 \Rightarrow \langle v\rangle \equiv \tfrac{\Delta x}{\Delta t} = v_0 + \tfrac{1}{2}a\Delta t$$

If the initial time is at t=0 and t_f=t, the average velocity is then:

$$\langle v\rangle = v_0 + \tfrac{1}{2}a\,t$$

Now, the average change in velocity in this situation is given by:

$$\langle \Delta v\rangle = \tfrac{1}{2}\Delta v$$

From above, we have:

$$\langle v\rangle = v_0 + \tfrac{1}{2}a\,t$$

$$v_f = v_0 + a\,t \text{ and } v_i = v_0 \Rightarrow \Delta v \equiv v_f - v_0 = a\,t = 2\langle \Delta v\rangle$$

$$\text{Thus,} \langle \Delta v\rangle = \tfrac{1}{2}[v_f - v_0]$$

Of course, there is also another pretty easy way to see this result. Consider the action of a velocity undergoing an average acceleration:

$$v = v_0 + at$$

Let's simply add the initial velocity to both sides and divide by 2:

$$v_0 + v = v_0 + v_0 + at \Rightarrow \tfrac{1}{2}[v_0 + v] = v_0 + \tfrac{1}{2}at \Rightarrow \langle v\rangle = v_0 + \tfrac{1}{2}at$$

Calculus Additional Section

The proof of the average velocity I used above is easily shown via calculus.
The average of a continuous function f(x) is more generally defined by:

$$\langle f(x)\rangle \equiv \frac{\int_{x=a}^{x=b} f(x)dx}{\int_{x=a}^{x=b} dx} = \frac{1}{b-a}\int_{x=a}^{x=b} f(x)dx$$

Here, if we look at the specific case of velocity and a constant acceleration, we then have:

$$f(x) \to v(t') = v_0 + a\,\Delta t'$$

On the interval between t'=0 and t'=t, we then have:

$$\langle v\rangle = \tfrac{1}{t}\left\{\int_{t'=0}^{t'=t} v_0\,dt + \int_{t'=0}^{t'=t} a\,t'dt\right\} = \tfrac{1}{t}\left\{v_0 t + \tfrac{1}{2}at^2\right\} = v_0 + \tfrac{1}{2}at$$

You will want to know all of the above equations for the case of a constant acceleration.
The calculus implies the following:

$$v = \frac{dx}{dt}$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$\Delta v = v(t) - v_0 = \int_{t=0}^{t_f} a(t)dt$$

$$\Delta x = x(t) - x_0 = \int_{t=0}^{t=t_f} v(t)dt = \int_{t=0}^{t=t_f} v_0 dt + \int_{t=0}^{t_f}\int_{t'=0}^{t}\left[a(t')dt'\right]dt$$ (the t' indicates a dummy variable).

You can (and should) easily verify that in the case of a constant acceleration, $a(t) = a$, the first three equations result.

	Physics 240: Worksheet 02	Name ______________

(1) Suppose a particle experiences an acceleration which varies as: $a(t) = bt$ where b is a positive constant and has SI units of m/s^3. Find the velocity and position of the particle at time t>0. Then show how, given this position, you can find the velocity and acceleration as a function of time.

(2) A sky diver with a parachute unopened, falls 500 m in 13.0 s. The sky diver opens a parachute and falls another 356 m in 142 s. What is the sky diver's average velocity (both magnitude and direction) the entire fall?

(3) Starting from rest, a cart reaches a speed of 4.1 m/s in 2.0 s. What is the cart's speed after an additional 3.0 s has elapsed, assuming the cart's acceleration remains the same?

(4) A soccer player starts from rest and sprints to a speed of 6.0 m/s in 1.5 s. Assuming that the player accelerates uniformly, determine the distance the player runs.

(5) The length of the barrel of a dart gun is 1.2 m. Upon leaving the barrel, a dart has a speed of 12 m/s. Assuming that the dart is uniformly accelerated, how long does it take for the dart to travel the length of the barrel?

(1) Suppose a particle experiences an acceleration which varies as: $a(t) = bt$ where b is a positive constant and has SI units of m/s^3. Find the velocity and position of the particle at time t>0. Then, show how, given this position, you can find the velocity and acceleration as a function of time.

$$a = \frac{dv}{dt} \Rightarrow a\ dt = dv \Rightarrow \int_{t'=0}^{t} (bt')dt' = \int_{v_0}^{v} dv$$

$$\Rightarrow b\frac{t'^2}{2}\Big|_0^t = v\Big|_{v_0}^{v} \Rightarrow b\frac{t^2}{2} = v - v_0$$

$$\Rightarrow v = v_0 + b\frac{t^2}{2}$$

We can now find the position at any later time by integrating this velocity.

$$v = \frac{dx}{dt} \Rightarrow dx = v\ dt$$

$$\Rightarrow \int_{x_0}^{x} dx = \int_{t=0}^{t} v_0 dt + \frac{b}{2}\int_{t=0}^{t} t^2 dt$$

$$\Rightarrow x - x_0 = v_0\ t\Big|_0^t + \frac{b}{2}\frac{t^3}{3}\Big|_0^t = v_0 t + \frac{bt^3}{6}$$

$$\Rightarrow x = x_0 + v_0 t + \frac{bt^3}{6}$$

On the other hand, suppose that the position as a function of time is given by:

$$x = x_0 + v_0 t + \frac{b}{6}t^3.$$

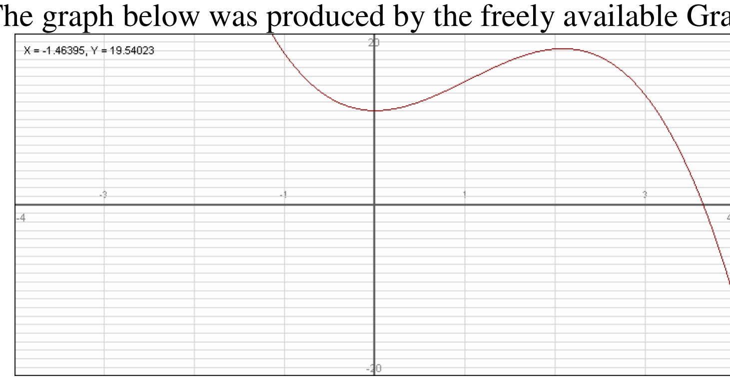
We can obtain the velocity and acceleration at all later times by taking derivatives:

$$v(t) = \frac{dx}{dt} = v_0 + \frac{bt^2}{2}$$

$$a(t) = \frac{dv}{dt} = bt$$

Now here is an additional calculus-type thing to think about: suppose as a function of time, the position of a particle is given by $x = 11 + 5t^2 - \frac{5}{3}t^3$.

What is the maximum x for positive time? Answer: take the derivative and solve for it being equal to zero: $\frac{dx}{dt} = +10t - 5t^2 = 0 \Rightarrow t = 0 \text{ or } t = \frac{10}{5} = 2$. For the first solution: x=11. For the second solution, x=17.7 which is the maximum value. You can, provided the minima or maxima exist, do similar things for the velocity and acceleration. However: be careful to not let the calculus obscure your understanding of the simple non-calculus versions also! The graph below was produced by the freely available Graphcal program.



(2) A sky diver with a parachute unopened, falls 500 m in 13.0 s. The sky diver opens a parachute and falls another 356 m in 142 s. What is the sky diver's average velocity (both magnitude and direction) the entire fall?

$\langle v \rangle = \frac{\Delta x}{\Delta t}$. Here, Δx=500 m + 356 m = 856 m. Δt=13.0 s + 142 s = 155 s

The average speed is then $\langle v \rangle = \frac{856m}{155s} = 5.52 m/s$. If we assume the direction of +y is up, then the vector average velocity is $\langle \vec{v} \rangle = 0\hat{i} - 5.52\hat{j}$ m/s

(3) Starting from rest, a cart reaches a speed of 4.1 m/s in 2.0 s. What is the cart's speed after an additional 3.0 s has elapsed, assuming the cart's acceleration remains the same?

First, we want to find the acceleration which is given by:

$$a = \frac{\Delta v}{\Delta t} = \frac{4.1 m/s}{2.0 s} = 2.1 \ \frac{m}{s^2}$$

Now, use this to find the velocity at any time.

$$v(t) = v_0 + at = 0 + (2.1)t = (2.1)5 = 10.5 \ \frac{m}{s}$$

Note that v_0 =0 since the cart starts from rest.

It is important to know and understand the meaning of v(t). This means the velocity as a function of t. It does not mean v times t. You will often see this type of notation in my class to designate that something is a function of something else.

(4) A soccer player starts from rest and sprints to a speed of 6.0 m/s in 1.5 s. Assuming that the player accelerates uniformly, determine the distance the player runs.
We need to find the acceleration first. This is given by:

$$a = \frac{\Delta v}{\Delta t} = \frac{6.0 m/s}{1.5 s} = 4.0 m/s^2$$

Here, however, we're really interested in $\Delta x = x_f - x_i$ (or $\Delta x = x_f - x_0$) since that gives the distance. Since the player starts from rest, v_0=0. The needed equation simplifies to become: $\Delta x = \frac{1}{2}at^2$ so the distance is $\Delta x = \frac{1}{2}(4.0)(1.5)^2 = 4.5m$

(5) The length of the barrel of a dart gun is 1.2 m. Upon leaving the barrel, a dart has a speed of 12 m/s. Assuming that the dart is uniformly accelerated, how long does it take for the dart to travel the length of the barrel?

method 1:

The easiest way to answer this question is this: the average velocity that the dart has is given by:

$$\langle v \rangle = v_0 + \tfrac{1}{2} a\, t$$

We also know the average velocity is 6 m/s since the system has a constant acceleration:

$$\langle v \rangle = \tfrac{1}{2}\left[v_f - v_0\right] = \tfrac{12}{2} = 6$$

The time the dart is inside the barrel is then given by the definition of average velocity:

$$\langle v \rangle = \tfrac{\Delta x}{\Delta t} \Rightarrow \Delta t = \tfrac{\Delta x}{\langle v \rangle} = \tfrac{1.2\,m}{6\,m/s} = 0.2\,s$$

I find this method to be less intuitive than the direct brute-force method which is below.

method 2:

We need the third equation (which eliminates time) to obtain the acceleration. This is given by: $v^2 = v_0^2 + 2a(\Delta x)$ We are thus able to obtain the acceleration:

$$a = \tfrac{v^2 - v_0^2}{2(\Delta x)} = \tfrac{(12)^2 - 0^2}{2(1.2)} = 60\ \tfrac{m}{s^2}$$

Now how long does it take for this to happen. There are two ways to answer this question.

(1) Use the second equation and solve for time: $v = at \Rightarrow t = \tfrac{v}{a} = \tfrac{12}{60} = 0.2$ s

(2) Use the first equation and solve for time:

$$x(t) = x_0 + v_0 t + \tfrac{1}{2} a_x t^2 \Rightarrow \left[\tfrac{1}{2} a_x\right] t^2 + \left[v_0\right] t + \left[x_0 - x\right] = 0 \Rightarrow \left[\tfrac{1}{2} a_x\right] t^2 + \left[v_0\right] t + \left[-\Delta x\right] = 0$$

This is a quadratic equation of the form: $ax^2 + bx + c = 0 \Rightarrow x = \tfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Here, we then have:

$$t = \frac{-v_0 \pm \sqrt{v_0^2 - 4\left[\tfrac{1}{2} a_x\right]\left[-\Delta x\right]}}{2\left[\tfrac{1}{2} a_x\right]} = \frac{\pm\sqrt{0 + 2(60)(1.2)}}{60} = \pm\frac{\sqrt{144}}{60} = \pm\frac{12}{60} = \pm 0.2\text{s}$$

The correct physical solution is t=0.2s (not negative). The second way only worked so easily because the initial velocity was zero. In general, in order to use the first equation, you would need to solve the quadratic equation.