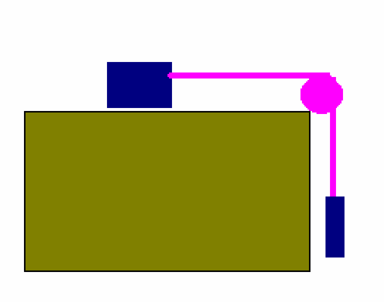
	Physics 240: Worksheet 10	Name _________________

(1) Hooke's law states that the force exerted by a spring is given by $\vec{F} = -k(\Delta\vec{x}) = -k(\vec{x}_f - \vec{x}_i)$. This means the more you push or pull on a spring, the more the spring presses or pulls back and in the opposite direction of the displacement of the spring. Find the work an external agent does to compress the spring through a displacement $\Delta\vec{x}$. Then find the work done by the spring in being compressed by the same displacement. Note: k is the spring constant measured in N/m in the SI system and is positive. You will measure the spring constant in lab 5.

(2) Suppose a bungee jumper[1] (m=100 kg) has bungee cords with a spring constant of 40 N/m. The bungee jumper jumps off of a very high bridge and falls for 20 m until the bungee cords start to stretch. How far from the point of the jump will the bungee jumper be when the bungee jumper finally stops.

[1] Don't try this.

(3) Two masses are arranged on a frictionless table as shown. When the second mass has fallen through a distance h, how fast is the system moving?



(4) Show the generalization of energy conservation to include non-conservative forces. Then if a mass is lying on a floor with a coefficient of friction μ is kicked so that it has an initial velocity v, how far will it go?

(5) Suppose a mass m slides down an inclined plane (of angle θ) with a coefficient of friction given by μ. How fast is the mass moving at the bottom of the plane if it falls through a vertical height y after being given a theoretical tiny push?

(1) Hooke's law states that the force exerted by a spring is given by

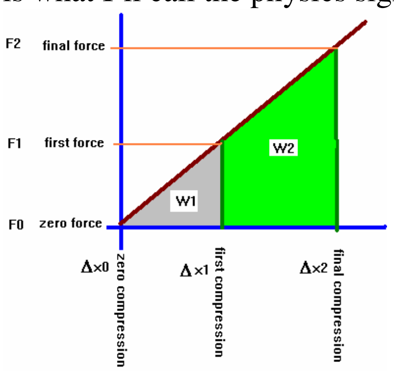
$$\vec{F} = -k\left(\Delta\vec{x}_{\text{spring}}\right) = -k\left(\Delta\vec{x}_{\text{final}_{\text{spring}}} - \Delta\vec{x}_{\text{initial}_{\text{spring}}}\right).$$

Find the work an external agent does to compress the spring through a displacement $\Delta\vec{x}$. Then find the work done by the spring in being compressed by the same displacement. Note: k is the spring constant measured in N/m in the SI system and is positive. You will measure the spring constant in lab 5.

I am trying to be very clear here that the coordinate that appears in Hooke's law really refers to spring compression or expansion and not position in space. The action of the force is such that the more you push or pull on a spring, the more the spring presses or pulls back and in the opposite direction of the displacement of the spring.

The following words are important and I have chosen them carefully.

There is something important to understand at the very beginning here: the work **done on a system and the work done by a system are two different things**. If work (W) is done on a system by an external agent, we would say that this work was positive (so long as F and Δs are in the same direction) from the point of view of the agent (the agent did work) and negative from the point of view of the system (the system had work done on it). This is what I'll call the physics sign convention.



We can determine the work done graphically because the work is given by the area under a plot of force-displacement graph as I am showing below. I should explain that what this plot shows is the amount of force that would be exerted at a certain value for the compression of the spring with the spring initially uncompressed.

Looking at the diagram, W_1 is the work required to compress the spring up to the amount of compression that I've called the "first compression." W_1 is equal to the area under the curve which is the grey shaded area. It is given by:

$$W_1 = \tfrac{1}{2}\left[\Delta x_1 - \Delta x_0\right]\left[F_1 - F_0\right]$$

But if the spring obeys Hooke's law then we have:

$$\left[F_1 - F_0\right] = F_1 = k\left[\Delta x_1 - \Delta x_0\right] = k\left(\Delta x_1\right)$$

So the quantity W_1 is given by:

$$W_1 = \tfrac{1}{2}k\left(\Delta x_1\right)^2$$

where I have assumed the x_0 and F_0 are both zero. Now to calculate the work required to compress the spring from X_1 to X_2, you will need again to calculate an area but this area is a bit more complicated here. If fact, we have:

$$W_2 = \left[\Delta x_2 - \Delta x_1\right]\left[F_1 - F_0\right] + \tfrac{1}{2}\left[\Delta x_2 - \Delta x_1\right]\left[F_2 - F_1\right] = kx_1\left[\Delta x_2 - \Delta x_1\right] + \tfrac{1}{2}k\left[\Delta x_2 - \Delta x_1\right]^2 = \tfrac{1}{2}k\left(\Delta x_2\right)^2 - \tfrac{1}{2}k\left(\Delta x_1\right)^2$$

The total work that the agent had to do to make the **initially uncompressed** spring have a compression given by Δx_2 is then:

$$W_{\substack{\text{by}\\\text{agent}\\\text{on}\\\text{spring}}} = \tfrac{1}{2}k\left(\Delta x_2\right)^2$$

Also notice: there is work done but no "change in work."

Now for the calculus version of this:

Our starting point is a statement without proof that the Hook's law force is conservative. We then have:

$$đW = \vec{F} \bullet d\vec{s}$$

I have defined the funny looking symbol for the differential for a very good reason: work is a funny kind of beast: it is not an exact differential mathematically. What that means for us in more realistic terms is this:

$$\int đW = W \text{ not } \cancel{\Delta W}$$

In physics, there is not an entity called "change in work" but there are small quantities of work, which added up give the total work.

When the agent compress the spring, the agent pushes the spring with a force that is, in fact, in the same direction as the displacement so the dot product is eliminated and the vectors are also eliminated. The total work done is given by:

$$W = \oint đW$$

Before I go further, let me explain that little circle: if you do not have a conservative force, then the work done will depend upon the path that you take. A frictional force is an example of a non-conservative force and this would depend upon the path taken. Without proof, however, I will state that Hook's law represents a conservative force, as are gravitational forces and electro**static** forces.

We thus have all we need to calculate the work done by the agent:

$$W_{\substack{\text{by}\\\text{agent}\\\text{on}\\\text{spring}}} = \oint đW = \oint_{s_1 \to s_2} \vec{F} \bullet d\vec{s} = \int_{s_1}^{s_2} \vec{F}_{\text{conservative}} \bullet d\vec{s} = \int_{\Delta x_1}^{\Delta x_2} k\left(\Delta\vec{x}_{\text{spring}}\right) d\left(\Delta\vec{x}_{\text{spring}}\right) = \tfrac{1}{2}k\left(\Delta\vec{x}_2\right)^2 - \tfrac{1}{2}k\left(\Delta\vec{x}_1\right)^2$$

Now, suppose a mass m is placed on the compressed spring. After the spring is released, how fast will the mass move when it passes through the equilibrium (uncompressed) position of the spring?

Answer: Apply conservation of energy: $\begin{matrix} U_i = \tfrac{1}{2}kx^2 & U_f = 0 \\ K_i = 0 & K_f = \tfrac{1}{2}mv^2 \end{matrix}$.

This gives: $U_i + K_i = U_f + K_f \Rightarrow \tfrac{1}{2}kx^2 + 0 = 0 + \tfrac{1}{2}mv^2 \Rightarrow v = \pm x\sqrt{\tfrac{k}{m}}$. Which of the signs is chosen depends upon the direction the spring was initially compressed.

As a modification, suppose the spring were pointed upward. What happens then? The answer is not so simple mathematically but can still be solved:

$$\tfrac{1}{2}kx^2 - mgx = \tfrac{1}{2}mv^2 \Rightarrow v = \pm\sqrt{\tfrac{k}{m}x^2 - 2gx}\ .$$

This is the velocity that the mass has at the instant it would leave the spring.

(2) Suppose a bungee jumper[1] (m=100 kg) has bungee cords with a spring constant of 40 N/m. The bungee jumper jumps off of a very high bridge and falls for 20 m until the bungee cords start to stretch. How far from the point of the jump will the bungee jumper be when the bungee jumper finally stops.

[1] Don't try this.

We have already seen that work done on a system can be calculated, and you need to be a little bit careful about specifying exactly what is doing the work in order to get the sign of the work represented correctly.

Energy is conserved. This means that $\Delta U_{gravitational} + \Delta U_{spring} + \Delta K = 0$. Notice that we are able to ignore the details of the velocity and kinetic energy here. In reality, wind resistance will slow the jumper until a "terminal velocity" is obtained (this is what happens to sky divers). On to the problem: we calculate each of the changes:

$$\Delta U_{gravitational} = mgy_f - mgy_i = mgy_f$$

Now the next thing to consider is this: the final position of the jumper does not correspond to the expansion of the cord. Since the coordinate in Hooke's law strictly refers to the spring expansion, we need to reflect this in the potential energy of the spring. Let us call the zero in potential energy the unstretched position of the cord. The initial position of the jumper is then +20.

Let us calculate each of the terms required for the energy equation.

(1) Since the jumper is initially at rest and at the end the jumper is at rest:

$$\Delta K = K_f - K_i = 0$$

(2) The change in gravitational potential energy is going to be given by:

$$\Delta U_g = mg\left(y_f\right) - mg\left(20\right)$$

where y_f is the final position of the mass.

(3) The change in potential energy of the spring is given by:

$$\Delta U_s = \tfrac{1}{2}k\left(y_f\right)^2 - 0 = \tfrac{1}{2}k\left(y_f\right)^2$$

Again, y_f is going to need to be a negative quantity as this problem is set up.

If we put everything together, we then have:

$$mgy_f - 20mg + \tfrac{1}{2}ky_f^2 = 0 \Rightarrow y_f^2 + \tfrac{2mg}{k}y - \tfrac{40mg}{k} = 0$$

mg/k=24.5

We can now use the particular numerical values for this problem:

$$y_f^2 + 49y_f - 980 = 0 \Rightarrow y_f = \frac{-49 \pm \sqrt{(49)^2 - 4(-980)}}{2} = \frac{-49 \pm \sqrt{2401+3920}}{2} = \frac{-49 \pm \sqrt{6321}}{2} = \frac{-49 \pm 79.5}{2} = \begin{matrix} 15.3\,m \\ -64.3\,m \end{matrix}$$

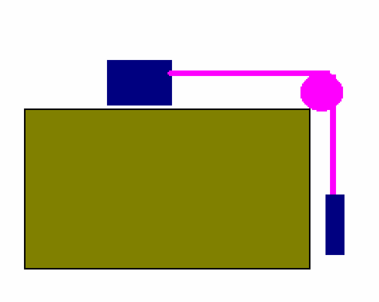
Now as the problem has been set up, the physically valid solution is the one with the negative coordinate so we have:

$$y_f = -64.3\,m$$

We can check our work by ignoring the initial 20 m. In this case, the original equation becomes $mgy + \tfrac{1}{2}ky^2 = 0 \Rightarrow y = -\tfrac{2mg}{k} = -\tfrac{1960}{40} = -49$ or 0.

In this case, the fact that y is negative is implied by our change in the gravitational potential. Then answer: the jumper stops 84 m below the bridge (don't forget the original 20). Whew!!

(3) Two masses are arranged on a frictionless table as shown. When the second mass has fallen through a distance h, how fast is the system moving?



The solution to this is shown as an animated gif. Please refer to that for the solution.

http://www.lyon.edu/webdata/users/shutton/animations/energy1.gif

However, I also now want to write out the solution.

Let's call the mass on top of the table m_1 and the hanging mass m_2. Energy is conserved so we have:

$$\Delta U_g + \Delta K = 0$$

We need to calculate each of the terms. I will call the y coordinate at the top of the table zero. Thus:

$$\begin{matrix} \Delta U_g = m_2 g y - 0 = -m_2 g h - 0 = -m_2 g h \\ \Delta K = \tfrac{1}{2}(m_1 + m_2)v^2 \end{matrix} \Rightarrow -m_2 g h + \tfrac{1}{2}(m_1 + m_2)v^2 \Rightarrow v = \pm\sqrt{\tfrac{2 m_2 g h}{(m_1 + m_2)}}$$

The particular sign for the velocity really depends upon the mass that you are asking about: for the first mass, positive. For the second mass, negative.

(4) Show the generalization of energy conservation to include non-conservative forces. Then if a mass is lying on a floor with a coefficient of friction μ is kicked so that it has an initial velocity v, how far will it go?

Solution:

The generalization of the equation expressing energy conservation is to include a term reflecting the loss of kinetic energy due to non-conservative forces, ΔK_{NC}. The more general form of the equation then becomes $\Delta K_{NC} = \Delta K + \Delta U$ where the subscript c means "conservative". The only trick to using this equation is to be able to calculate ΔK_{NC}. This is calculated from the work-energy theorem:

$$\Delta K_{NC} = \text{work}_{\substack{\text{by}\\ \text{system}\\ \text{against}\\ \text{NonConservative}\\ \text{forces}}}$$

Since the only non-conservative force we'll be using in this course is friction, we can say that mostly for us, this calculation reduces to $\Delta K_{NC} = \vec{f} \bullet \vec{x}$. Let's see how to do this for the given example:

$$\Delta U = U_f - U_i = 0$$

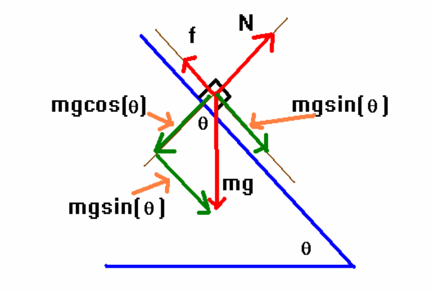
$$\Delta K = K_f - K_i = 0 - \tfrac{1}{2} mv^2$$

$$\Delta K_{NC} = \vec{f} \bullet \vec{x} = -\mu mgx$$

We put this together to get:

$$-\mu mgx = -\tfrac{1}{2} mv^2 \Rightarrow x = \tfrac{v^2}{2\mu g}\ .$$

(5) Suppose a mass slides down an inclined plane (of angle θ) with a coefficient of friction given by μ. How fast is the mass moving at the bottom of the plane if it falls through a vertical height h after being given a theoretical tiny push?



Solution: It is best to draw a free body diagram here because you need to find the frictional force. If you do this, you will find

$$N = mg\cos(\theta)$$

so that the frictional force is given by

$$\vec{f} = -\mu mg\cos(\theta)\hat{i}$$

where I have rotated coordinates down the plane. The distance the mass moves down the plane is related to the distance down the plane (x) by:

$$h = x\sin(\theta)$$

so that

$$x = \frac{h}{\sin(\theta)}.$$

The work against friction is given by

$$\Delta K_{NC} = -\mu N x = -\mu \frac{mg\cos(\theta)h}{\sin(\theta)} = -\mu mgh\cot(\theta).$$

The change in potential energy is

$$\Delta U = U_f - U_i = -mgh$$

and

$$\Delta K = K_f - K_i = \tfrac{1}{2}mv^2$$

We put all this together:

$$\Delta K_{NC} = \Delta K + \Delta U \Rightarrow -\mu mgh\cot(\theta) = -mgh + \tfrac{1}{2}mv^2.$$

We can simplify and solve this for v:

$$-\mu gh\cot(\theta) = -gh + \tfrac{1}{2}v^2 \Rightarrow \tfrac{1}{2}v^2 = gh(1-\mu\cot(\theta)).$$

If we carry this a bit further, we find the solution for v:

$$v = \sqrt{2gh(1-\mu\cot(\theta))}.$$

You might wonder how careful you need to be when making up problems like this. Look at the thing under the square root: if μ=0, we are in the frictionless case and v correctly reproduces that for free-fall (you ought to be able to show this yourself please try!). Problems will happen when $1-\mu\cot(\theta) = 0 \Rightarrow \cot(\theta) = \frac{1}{\mu}$ or tan(θ)=μ. You've seen this before! For tan(θ)<μ, the problem won't work since the mass won't slide with a constant velocity.