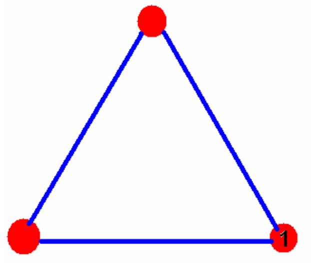
	Physics 250: Worksheet 02	Name _______________

Concepts: Electric Field, lines of force, charge density, dipole moment, electric dipole

(1) An equilateral triangle with each side of length 0.10 m has identical charges of $+q=1.0\mu C$. What is the net electrostatic **vector** force on charge 1?



(2) A point charge $q_1=-3.00\mu C$ is located at x=0. A second charge $q_2=+6.00\mu C$ is located at x=1.00 m. Find a point other than infinity where the electric field is zero.

(3) The electric dipole consists of a positive and a negative charge separated by a distance of 2a. Suppose in this case, your dipole had +q at x=a and -q at x=-a. Find an expression for the electric field along the y-axis. You should then be able to show that the electric field behaves as $\mathbf{E_x \approx 2kqa/y^3}$ at distant points along the y-axis.

(4) Suppose in this case, your dipole had +q at x=a and -q at x=-a. Find an expression for the electric field along *the x-axis* at x>a. You should then be able to show that the electric field behaves as $\mathbf{E_x \approx 4kqa/x^3}$ at distant points along the *x* -axis. Then write the result in terms of the dipole moment.

(5) Suppose that you have a ring of radius r=a and total charge Q located in the x-y plane. What is the electric field for points along the symmetry axis of this ring? How does this field behave along the axis at distant points *along the symmetry axis*?

We have previously defined the electric force between two charges and we have talked about the law of charges. It turns out that it is less important to talk about the electric force than another quantity called the electric field. The **electric field** is a real physical entity and carries away energy from an accelerating electric charge.

The electric field is defined by:

$$\vec{E} \equiv \frac{\vec{F}_{electric}}{q}$$

Here, by definition, q is a **positive** test charge.

E points in the direction that a **positive test charge** would move under the influence of an electric force. These "lines of force" can be sketched with a few rules:
(1) They point away from positive charges.
(2) They point towards negative charges.
(3) They don't intersect.
(4) They point normal to the surface of a conductor.
(5) The density of these lines is an indicator of the electric field strength
(6) A positive charge placed on one of the field lines accelerates in the line's direction.

I'll later show you how to draw these lines of force.
In more elegant terms, then, using the definition of force that we had in the last lecture, we can write the electric field as:

$$\vec{E}_p = \sum_{i=1}^{n} k \frac{q_i}{\left|\vec{r}_p - \vec{r}_i\right|^2}\hat{r}_{ip} = k\sum_{i=1}^{n} \frac{q_i}{\left|\vec{r}_{ip}\right|^2}\hat{r}_{ip}$$

Let's look at each of the symbols:
"n"=# of discrete charges in the system
q_i is the i^{th} charge in the system.
"k" is coulomb's constant

$\vec{r}_p$ is the vector from the origin pointed towards the point p in space. This would also be the location of the positive test charge so the notation that we have developed is really the same here: the test charge is now charge p. $\vec{r}_i$ is the vector from the origin pointed towards the charge q_i in space. $\hat{r}_{ip}$ is the unit vector directed from the charge q_i towards the point p in space. Don't get hungup on the fact that a particular charge might not be located at the origin: apply the rules I've shown you in worksheet 1 and you will correctly calculate the electric field.

One thing that you want to know is just how do you calculate $\hat{r}_{ip}$. Here is the way although you have already seen this in worksheet 1. Firstly, $\vec{r}_{ip} \equiv \vec{r}_p - \vec{r}_i$. We can now, from this find the unit vector pretty easily. Again, in words: $\vec{r}_{ip}$ is the vector pointing from charge i toward point p in space. The unit vector pointing in this direction is given by:

$$\hat{r}_{ip} = \frac{\vec{r}_{ip}}{\left|\vec{r}_p - \vec{r}_i\right|}$$

So, let me show you an example here:
Suppose:

$$\vec{r}_p = x_p\hat{x} + y_p\hat{y} + z_p\hat{z} \text{ and } \vec{r}_i = x_i\hat{x} + y_i\hat{y} + z_i\hat{z}.$$

Then,

$$\hat{r}_{ip} = \frac{(x_p - x_i)\hat{x} + (y_p - y_i)\hat{y} + (z_p - z_i)\hat{z}}{\sqrt{(x_p - x_i)^2 + (y_p - y_i)^2 + (z_p - z_i)^2}}$$

Here are some numerical examples:
Suppose $\vec{r}_i = 1\hat{x} + 2\hat{y} + 3\hat{z}$ and $\vec{r}_p = 3\hat{x} + 2\hat{y} + 1\hat{z}$. Then:

$$\vec{r}_{ip} = \vec{r}_p - \vec{r}_i = (3-1)\hat{x} + (2-2)\hat{y} + (1-3)\hat{z} = 2\hat{x} + 0\hat{y} - 2\hat{z}$$

The unit vector is then:

$$\hat{r}_{ip} = \frac{2\hat{x} - 2\hat{z}}{\sqrt{2^2 + 2^2}} = \frac{2}{\sqrt{8}}(\hat{x} - \hat{z}) = \frac{1}{\sqrt{2}}(\hat{x} - \hat{z})$$

Another really easy example: Suppose $\vec{r}_i = 1\hat{x} + 0\hat{y} + 0\hat{z}$ and $\vec{r}_p = 0\hat{x} + 0\hat{y} + 0\hat{z}$.

$$\vec{r}_{ip} = (0-1)\hat{x} = -\hat{x} \text{ and } \hat{r}_{ip} = \frac{-\hat{x}}{1} = -\hat{x}$$

Notice that $|\vec{r}_p - \vec{r}_i|$ is simply the distance between the charge and the point p.
One notational detail: I'll write: $\vec{r}_{ip} \equiv \vec{r}_p - \vec{r}_i$ occasionally and you'll probably do the same.

While this is more technical, in principle you might just find it an easier approach than having to resolve electric field components each time. This applies to charges that are discrete.

For the calculus people, we have a more general definition which treats charges as a continuum:

$$\vec{E}_p = \int_{\substack{\text{all} \\ \text{charges}}} k \frac{dq_i}{(\vec{r}_p - \vec{r}_i)^2} \hat{r}_{ip}$$

(Notice that I've retained the redundant "i" subscript here to make sure you know that we're talking about charges and that's what is being integrated over).

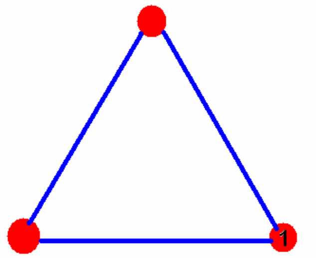
Where the unit vector symbol really has the meaning that it is the unit vector directed from the charge point dq towards the point p in space. I encourage you to retain this notation in all your work in order to assure yourself that you know what is happening.

One final important point. I'll introduce in problems 3 and 4 the electric dipole moment. For a collection of j charges, we define the dipole moment as:

$$\vec{p} = \sum_{j=1}^{n} q_j \vec{r}_j$$

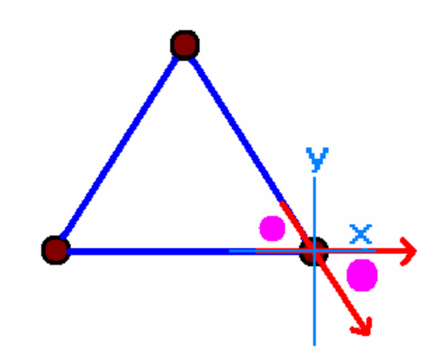
It's important not to confuse this "p" with the "p" which I'm using to designate the point in space. There is also one additional term which is going to be introduced later, namely the polarization of a material which is designated by P.

(1) An equilateral triangle with each side of length 0.10 m has identical charges of $+q=1.0\mu C$. What is the net electrostatic **vector** force on charge 1?



Solution: I want to show you 2 ways to do this problem. The first uses symmetry and is quicker. The second is the brute force method.

The force on any single charge is shown in red below:



The little purple dots indicate the same angle which is 180/3=60 degrees. This is the angle θ that I'm using below. This only works because each of the charges is the same and each distance is the same.
I am indicating forces with the red arrows.
Thus, the "off-axis" force is going to be given by:

$$\vec{F} = F_x\hat{x} + F_y\hat{y} = F\left(\cos(\theta)\hat{x} - \sin(\theta)\hat{y}\right)$$

The total force on any single charge is then given by:

$$\vec{F}_{net} = F\hat{x} + F_x\hat{x} + F_y\hat{y} = F(\left[1+\cos(\theta)\right]\hat{x} - \sin(\theta)\hat{y})$$

We find the magnitude of this force from Coulomb's law:

$$\left|\vec{F}\right| = \left|k\tfrac{q_1q_2}{r^2}\right| = 8.99x10^9\ \left(\tfrac{1x10^{-12}}{0.01}\right) = 0.899N$$

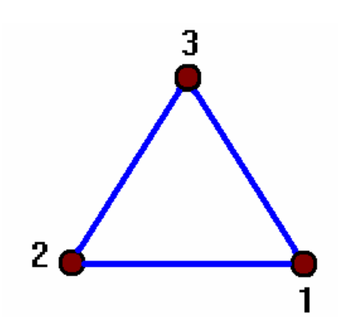
We can now find the force on this charge: $\vec{F} = 1.349\hat{x} - 0.779\hat{y}$ N

The magnitude of this force is $\left|\vec{F}\right| = \sqrt{1.349^2 + 0.799^2} = 1.558N$

The angle which this force makes with respect to the x-axis is given by:

$$\tan(\varphi) = \tfrac{F_y}{F_x} = \tfrac{-.779}{1.349} \Rightarrow \varphi = \begin{cases} -30 \\ 180+(-30) \end{cases} = \begin{matrix} -30^0 \\ 150^0 \end{matrix}$$

The correct result here is -30 degrees.



Now let me show you the second (and, in my opinion the more powerful) way to do this.

I am putting labels on the charges as shown. I will let charge 1 be at the origin, so it has coordinates (0,0).

Probably the hardest part is to find the coordinates of charge #3. However you want to add it up, the x coordinate of the charge is 0.05 m. The length of this vector pointing to charge 2 is 0.1m. Thus, the y coordinate is:

$$|\vec{r}_3|^2 = x_3^2 + y_3^2 \Rightarrow y_3^2 = |\vec{r}_3|^2 - x_3^2 \Rightarrow y_3 = \pm\sqrt{|\vec{r}_3|^2 - x_3^2}$$

$$y_3 = \sqrt{.1^2 - (-0.05)^2} = \sqrt{0.01 - 0.0025} = 0.0867$$

The various vectors are then:

$$\vec{r}_1 = 0\hat{x} + 0\hat{y};\ \vec{r}_3 = -0.05\hat{x} + 0.08667\hat{y};\ \vec{r}_2\text{=-0.1}\hat{x}\text{+0}\hat{y}\text{:}\vec{r}_p = 0\hat{x} + 0\hat{y}$$

Now we're going to need to calculate this:

$$\vec{F}_p = \sum_{\substack{i=1 \\ i\neq p}}^{n} k \frac{q_i q_p}{|\vec{r}_{ip}|^2}\hat{r}_{ip}$$

That means if we're calculating the force on charge 1, we need the following:

$$\vec{F}_{(p=1)} = \sum_{\substack{i=1 \\ i\neq(p=1)}}^{n} k \frac{q_i q_{(p=1)}}{|\vec{r}_{i(p=1)}|^2}\hat{r}_{i(p=1)} = k\frac{q_2 q_1}{|\vec{r}_{21}|^2}\hat{r}_{21} + k\frac{q_3 q_1}{|\vec{r}_{31}|^2}\hat{r}_{31}$$

We are going to need to calculate the various vectors involved here. I'm going to try to show this in excruciating detail here.

$$\vec{r}_{21} = \vec{r}_1 - \vec{r}_2 = [0\hat{x} + 0\hat{y}] - [-1\hat{x} + 0\hat{y}] = [0+1]\hat{x} = \hat{x}$$

$$\hat{r}_{21} = \frac{\vec{r}_{21}}{|\vec{r}_{21}|} = \frac{\hat{x}}{|\hat{x}|} = \hat{x}$$

$$\vec{r}_{31} = \vec{r}_1 - \vec{r}_3 = [0\hat{x} + 0\hat{y}] - [-0.05\hat{x} + 0.08667\hat{y}] = 0.05\hat{x} - 0.08667\hat{y}$$

$$\hat{r}_{31} = \frac{\vec{r}_{31}}{|\vec{r}_{31}|} = \frac{0.05\hat{x} - 0.08667\hat{y}}{\sqrt{(-0.05)^2 + (0.08667)^2}} = \frac{0.05\hat{x} - 0.08667\hat{y}}{.1} = 0.5\hat{x} - 0.8667\hat{y} = 0.5\hat{x} - \frac{\sqrt{3}}{2}\hat{y}$$

The electric field at charge 1 ("p") due to the other two charges is then:
We now have everything we need.

$$\vec{F}_1 = kq_1\left[\frac{q_2}{.1^2}\hat{x} + \frac{q_3}{.1^2}\left[0.5\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\right]\right]$$

In this particular problem each of the charges is the same. I'll let q represent this charge.

$$\vec{F}_1 = kq^2\left[\frac{1}{.1^2}\hat{x} + \frac{1}{.1^2}\left[0.5\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\right]\right]$$

$$\vec{F}_1 = \frac{kq^2}{.01}\left[\hat{x} + \left[0.5\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\right]\right] = \frac{kq^2}{.01}\left[1.5\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\right]$$

Since k=8.99x10^9 and q=1μC, we thus have:

$$\vec{F}_1 = \frac{8.99\times10^{9-12}}{.01}\left[1.5\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\right] = \frac{8.99\times10^{-3}}{.01}\left[1.5\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\right]$$

$$= 0.899\left[1.5\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\right] = 1.349\hat{x} - 0.7786\hat{y}$$

$$|\vec{F}_1| = \sqrt{(1.349)^2 + (-0.7786)^2} = 1.558\text{N}$$

The angle that the force makes with respect to the +x axis is

$$\tan(\varphi) = \frac{F_{1,y}}{F_{1,x}} = \frac{-0.7786}{1.349} = -0.577 \Rightarrow \varphi = -30^0$$

We could have found the same result by calculating the electric field at charge 1 due to charges 2 and 3. The electric field at this point is given by:

$$\vec{E}_p = \sum_{\substack{i=1 \\ (i \neq p)}}^{n} k \frac{q_i}{\left|\vec{r}_p - \vec{r}_i\right|^2} \hat{r}_{ip} = \sum_{\substack{i=1 \\ (i \neq p)}}^{n} k \frac{q_i}{\left|\vec{r}_{ip}\right|^2} \hat{r}_{ip}$$

this would give:

$$\vec{E}_p = \tfrac{1}{q}\left[1.349\hat{x} - 0.7786\hat{y}\right]; q = 1\times10^{-6}\,C$$

Then to get the force, multiply the electric field by the charge at point 1.

(2) A point charge q_1=-3.00μC is located at x=0. A second charge q_2=+6.00μC is located at x=1.00 m. Find a point other than infinity where the electric field is zero.

The electric field is defined by:

$$\vec{E}_p = \sum_{i=1}^{n} k \frac{q_i}{\left|\vec{r}_p - \vec{r}_i\right|^2} \hat{r}_{ip}$$

where p represents a point in space.

We locate the initial charge at $\vec{r}_1 = 0\hat{x}$ and the second charge at $\vec{r}_2 = 1\hat{x}$.

The vector pointing to p is given (in two dimensions) by:

$$\vec{r}_p = x_p\hat{x} + y_p\hat{y}$$

The electric field at any point in 2-D space is then given by:

$$\vec{E}_p = k\frac{q_1}{\left|\vec{r}_{1p}\right|^2}\hat{r}_{1p} + k\frac{q_2}{\left|\vec{r}_{2p}\right|^2}\hat{r}_{2p}$$

We can easily form each of these vectors now.

$$\vec{r}_{1p} = \vec{r}_p - \vec{r}_1 = \left[x_p\hat{x} + y_p\hat{y}\right] - \left[0\hat{x} + 0\hat{y}\right] = \left[x_p\hat{x} + y_p\hat{y}\right]$$

$$\hat{r}_{1p} = \frac{x_p\hat{x} + y_p\hat{y}}{\sqrt{x_p^2 + y_p^2}}$$

$$\vec{r}_{2p} = \vec{r}_p - \vec{r}_2 = \left[x_p\hat{x} + y_p\hat{y}\right] - \left[1\hat{x} + 0\hat{y}\right] = \left(x_p - 1\right)\hat{x} + y_p\hat{y}$$

$$\hat{r}_{2p} = \frac{\left(x_p - 1\right)\hat{x} + y_p\hat{y}}{\sqrt{\left(x_p - 1\right)^2 + y_p^2}}$$

We now form the electric field:

$$\vec{E}_p = kq_1\frac{1}{\left(x_p^2 + y_p^2\right)}\frac{x_p\hat{x} + y_p\hat{y}}{\sqrt{x_p^2 + y_p^2}} + kq_2\frac{1}{\left(\left(x_p - 1\right)^2 + y_p^2\right)}\frac{\left(x_p - 1\right)\hat{x} + y_p\hat{y}}{\sqrt{\left(x_p - 1\right)^2 + y_p^2}}$$

We need to solve this for $\vec{E} = \vec{0}$

In this particular problem, we notice q_2=-2q_1. Thus:

$$q_1\frac{1}{x_p^2 + y_p^2}\frac{x_p\hat{x} + y_p\hat{y}}{\sqrt{x_p^2 + y_p^2}} - 2q_1\frac{1}{\left(x_p - 1\right)^2 + y_p^2}\frac{\left(x_p - 1\right)\hat{x} + y_p\hat{y}}{\sqrt{\left(x_p - 1\right)^2 + y_p^2}} = \vec{0}$$

Now I'll show the algebraic steps.

$$\frac{1}{x_p^2 + y_p^2}\frac{x_p\hat{x} + y_p\hat{y}}{\sqrt{x_p^2 + y_p^2}} = 2\frac{1}{\left(x_p - 1\right)^2 + y_p^2}\frac{\left(x_p - 1\right)\hat{x} + y_p\hat{y}}{\sqrt{\left(x_p - 1\right)^2 + y_p^2}}$$

$$x: \frac{x_p}{\left[x_p^2 + y_p^2\right]^{3/2}} = \frac{2\left(x_p - 1\right)}{\left[\left(x_p - 1\right)^2 + y_p^2\right]^{3/2}}$$

$$y: \frac{y_p}{\left[x_p^2 + y_p^2\right]^{3/2}} = \frac{2y_p}{\left[\left(x_p - 1\right)^2 + y_p^2\right]^{3/2}}$$

It is easy to see that the y equation is satisfied with y_p=0. This could have also come from symmetry (in past versions of this problem I merely assumed this).

Use this in the x-equation:

$$\frac{x_p}{x_p^3} = \frac{2\left(x_p - 1\right)}{\left(x_p - 1\right)^3} \Rightarrow \frac{1}{x_p^2} = \frac{2}{\left(x_p - 1\right)^2} \Rightarrow x_p^2 = \tfrac{1}{2}\left(x_p - 1\right)^2 \Rightarrow 2x_p^2 = \left(x_p - 1\right)^2$$

$$\Rightarrow \pm\sqrt{2}x_p = x_p - 1 \Rightarrow 0 = x_p\left(1 \pm \sqrt{2}\right) - 1 \Rightarrow x_p\left(1 \pm \sqrt{2}\right) = 1$$

$$\therefore x_p = \frac{1}{1 \pm \sqrt{2}}$$

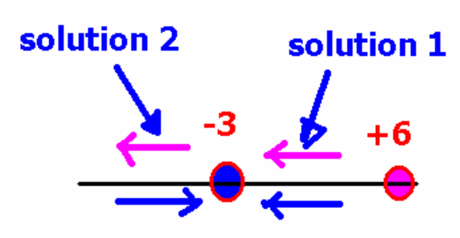
The two solutions are thus obtained. Let me confirm that E is zero for each of these points. If it's not we'll need to discard one solution.

On to the actual answer for the solutions:

if: $x_p = \frac{1}{1+\sqrt{2}} = 0.4142$, the electric field is not zero.

if: $x_p = \frac{1}{1-\sqrt{2}} = -2.4142$ the electric field is zero.

You can check these results using the spreadsheet on our website by choosing a test charge of +1 located at each of the two solutions for x. I have provided the spreadsheet showing the "zero" solution for you.



What this sketch above shows is that between the two charges, it is impossible to have zero electric field (solution 1 is between the two charges). Solution 2 however is in a region where it is possible to have zero electric field. The only other region where you might consider is outside (at positive x). However, you will never get zero there because the +6 charge is not only larger than the -3 charge, but it is also always closer to the point in that region.

Now, how else could this be solved? This is a quicker (and not nearly so mathematically clean here): You could say this: the electric field at a point along the x-axis is:

$$\vec{E}_p = k\frac{q_1}{x_{1p}^2}\hat{r}_{1p} + k\frac{q_2}{x_{2p}^2}\hat{r}_{2p}$$

Now let's loose the vector notation. This will increase the number of incorrect solutions but that's the price you'll need to pay for this. Thus, we might have this:

$\frac{q_1}{x_{1p}^2} - \frac{q_2}{x_{2p}^2} = 0$ or $\frac{q_1}{x_{1p}^2} + \frac{q_2}{x_{2p}^2} = 0$ here, q2=-2q1 so this gives: $\frac{1}{x_{1p}^2} + \frac{2}{x_{2p}^2} = 0$ or $\frac{1}{x_{1p}^2} - \frac{2}{x_{2p}^2} = 0$

Solving this then gives:

$$\frac{1}{x_{1p}^2} + \frac{2}{x_{2p}^2} = 0 \Rightarrow \frac{1}{x_{1p}^2} = -\frac{2}{x_{2p}^2} \Rightarrow x_{1p}^2 = -\frac{1}{2}x_{2p}^2 \Rightarrow x_{1p} = \pm\sqrt{-\frac{1}{2}}x_{2p}$$

$$\text{or } \frac{1}{x_{1p}^2} - \frac{2}{x_{2p}^2} = 0 \Rightarrow x_{1p} = \pm\sqrt{\frac{1}{2}}x_{2p}$$

Ok, so we at least know the relationship required for the two distances. We can now find possible solutions. Thus

$$|x-0| = \pm\sqrt{\tfrac{1}{2}}|x-1| \Rightarrow |x-0| = \sqrt{\tfrac{1}{2}}|x-1| \Rightarrow |x| = \sqrt{\tfrac{1}{2}}|x-1|$$

The solution here is then the following:

(a) if x<0 then:

$$-x = -\sqrt{\tfrac{1}{2}}(x+1) \Rightarrow x\left[-1+\sqrt{\tfrac{1}{2}}\right] = -\sqrt{\tfrac{1}{2}} \Rightarrow x = \frac{-\sqrt{\frac{1}{2}}}{\left[-1+\sqrt{\frac{1}{2}}\right]} = \frac{-1}{-\sqrt{2}+1} = \frac{1}{1-\sqrt{2}}$$

(b) if x>0 and x<1 then:

$$x = \sqrt{\tfrac{1}{2}}(1-x) \Rightarrow x\left[1+\sqrt{\tfrac{1}{2}}\right] = \sqrt{\tfrac{1}{2}} \Rightarrow x = \frac{\sqrt{\frac{1}{2}}}{\left[1+\sqrt{\frac{1}{2}}\right]} = \frac{1}{1+\sqrt{2}}$$

(c) if x>1 then

$$x = \sqrt{\tfrac{1}{2}}(x-1) \Rightarrow x\left(1-\sqrt{\tfrac{1}{2}}\right) = -\sqrt{\tfrac{1}{2}} \Rightarrow x = \frac{-\sqrt{\frac{1}{2}}}{\left(1-\sqrt{\frac{1}{2}}\right)} = \frac{-1}{\sqrt{2}-1} = \frac{1}{1-\sqrt{2}}$$

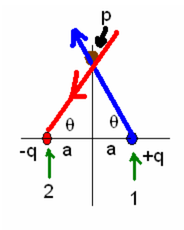
which is not less than 1.

Solutions (a) and (b) need to be tested with the same results as earlier.

The following problem is quite important. Be sure you understand it.

(3) The electric dipole consists of a positive and a negative charge separated by a distance of 2a. Suppose in this case, your dipole had +q at x=a and -q at x=-a. Find an expression for the electric field along the y-axis. You should then be able to show that the electric field behaves as $\mathbf{E_x \approx 2kqa/y^3}$ at distant points along the y-axis.

We begin with the definition of the electric field:



$$\vec{E}_p = \sum_{i=1}^{n} k \frac{q_i}{\left|\vec{r}_p - \vec{r}_i\right|^2} \hat{r}_{ip}$$

Now we need to obtain the various vectors involved.

$$\vec{r}_1 = a\hat{x} : \vec{r}_2 = -a\hat{x} : \vec{r}_p = y_p\hat{y}$$

$$\vec{r}_{1p} = -a\hat{x} + y_p\hat{y} : \vec{r}_{2p} = a\hat{x} + y_p\hat{y}$$

$$\hat{r}_{1p} = \frac{-a\hat{x}+y_p\hat{y}}{\sqrt{a^2+y_p^2}} : \hat{r}_{2p} = \frac{a\hat{x}+y_p\hat{y}}{\sqrt{a^2+y_p^2}}$$

Now we need to use these in the definition of the electric field.

$$\vec{E}_p = k\frac{q_1}{\left|\vec{r}_{1p}\right|^2}\hat{r}_{1p} + k\frac{q_2}{\left|\vec{r}_{2p}\right|^2}\hat{r}_{2p}$$

We thus have: (letting q_1 be the same magnitude as q_2):

$$\vec{E}_p = k\frac{q}{a^2+y_p^2}\frac{-a\hat{x}+y_p\hat{y}}{\sqrt{a^2+y_p^2}} - k\frac{q}{a^2+y_p^2}\frac{a\hat{x}+y_p\hat{y}}{\sqrt{a^2+y_p^2}} = \frac{kq}{\left[a^2+y_p^2\right]^{3/2}}\left\{\left(-a-a\right)\hat{x} + \left(y_p - y_p\right)\hat{y}\right\} \Rightarrow \vec{E}_p = -\frac{2akq}{\left[a^2+y_p^2\right]^{3/2}}\hat{x}$$

This is the actual answer. Now let's look at how this behaves for y>>a. The binomial expansion says:

$$\left(a^2+y_p^2\right)^{3/2} = y_p^3\left(\left(\frac{a}{y_p}\right)^2+1\right)^{3/2} \approx y_p^3\left(1+\frac{3}{2}\left(\frac{a}{y_p}\right)^2+\ldots\right)$$

Thus to first order, (if y>>a) we have:

$$\left(a^2+y_p^2\right)^{3/2} \approx \left|y_p\right|^3$$

The electric field at large distances along the perpendicular bisector of the dipole is:

$$\vec{E}_p \approx \frac{-2akq}{\left|y_p\right|^3}\hat{x}$$

Both of these results are extremely important for systems involving electric dipoles! It is also indeed very interesting to see that the dipole falls off as $1/y^3$ at large distances. The term p=2qa is called the magnitude of the electric dipole moment. We calculate the dipole moment for 2 equal and opposite charges as:

$$\vec{p} = \sum_{j=1}^{2} q_j\vec{r}_j = qa\hat{x} - q(-a\hat{x}) = q\left(2a\hat{x}\right) = q\vec{d}$$

where $\vec{d}$ is the vector pointing ***from the negative charge towards the positive charge***.

For a continuous charge distribution, the electric dipole moment is calculated as:

$$\vec{p} = \int_{\text{all charges}} \vec{r}_i\,\rho\left(\vec{r}_i\right)d^3r_i$$

In terms of the electric dipole moment at large distances along the symmetry axis we have (using only most significant terms):

$$\vec{E}_p = -k\frac{\vec{p}}{|y_p|^3}.$$

The following shows how this conclusion is obtained (this is for interested students):

Suppose you are off the symmetry axis:

$$\vec{r}_p = x_p\hat{x} + y_p\hat{y}$$

Then the calculation is a little bit more complicated: I am showing you the steps for future reference (the electric physical dipole is quite important for chemists).

$$\vec{r}_1 = a\hat{x} : \vec{r}_2 = -a\hat{x} : \vec{r}_p = x_p\hat{x} + y_p\hat{y}$$

$$\vec{r}_{1p} = \left(x_p - a\right)\hat{x} + y_p\hat{y} : \vec{r}_{2p} = \left(x_p + a\right)\hat{x} + y_p\hat{y}$$

$$\hat{r}_{1p} = \frac{\left(x_p - a\right)\hat{x} + y_p\hat{y}}{\sqrt{\left(x_p - a\right)^2 + y_p^2}} : \hat{r}_{2p} = \frac{\left(x_p + a\right)\hat{x} + y_p\hat{y}}{\sqrt{\left(x_p + a\right)^2 + y_p^2}}$$

The electric field is then:

$$\vec{E}_p = k\frac{q_1}{|\vec{r}_{1p}|^2}\hat{r}_{1p} + k\frac{q_2}{|\vec{r}_{2p}|^2}\hat{r}_{2p}$$

So:

$$\vec{E}_p = kq\left[\frac{\left(x_p - a\right)\hat{x} + y_p\hat{y}}{\left[\left(x_p - a\right)^2 + y_p^2\right]^{3/2}} - \frac{\left(x_p + a\right)\hat{x} + y_p\hat{y}}{\left[\left(x_p + a\right)^2 + y_p^2\right]^{3/2}}\right]$$

In the following, let $\vec{a} \equiv a\hat{x}$. Then:

$$\left[\left(x_p \pm a\right)^2 + y_p^2\right] = \left[x_p^2 + a^2 \pm 2x_p a + y_p^2\right] = \left[\vec{r}_p^2 + \vec{a}^2 \pm 2\vec{r}_p \bullet \vec{a}\right] = \left[\vec{r}_p^2 + \vec{a}^2 \pm 2|\vec{r}_p||\vec{a}|\cos(\theta)\right]$$

(Here, the angle is with respect to the +x axis).

let's look at the approximations for r>>a.

$$\left[\vec{r}_p^2 + \vec{a}^2 \pm 2|\vec{r}_p||\vec{a}|\cos(\theta)\right]^{-3/2} = |\vec{r}_p^2|^{-3/2}\left[1 + \frac{\vec{a}^2 \pm 2|\vec{r}_p||\vec{a}|\cos(\theta)}{\vec{r}_p^2}\right]^{-3/2} \approx |\vec{r}_p|^{-3}\left[1 \mp 3\frac{|\vec{r}_p||\vec{a}|\cos(\theta)}{\vec{r}_p^2}\right] \approx |\vec{r}_p|^{-3}\left[1 \mp 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^2}\right]$$

So in the expression for E above, we'll replace:

$$\frac{1}{\left[\left(x_p \pm a\right)^2 + y_p^2\right]^{3/2}} \approx \frac{1}{|\vec{r}_p|^3}\left[1 \mp 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^2}\right]$$

We then have:

$$\vec{E}_p \approx \frac{kq}{|\vec{r}_p|^3}\left[\left[\left(x_p - a\right)\hat{x} + y_p\hat{y}\right]\left[1 + 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^2}\right] - \left[\left(x_p + a\right)\hat{x} + y_p\hat{y}\right]\left[1 - 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^2}\right]\right]$$

Simplifying:

$$\vec{E}_p \approx \frac{kq}{|\vec{r}_p|^3}\begin{bmatrix} x_p\hat{x}\left[+\left[1 + 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^2}\right] - \left[1 - 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^2}\right]\right] \\ +a\hat{x}\left[-\left[1 + 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^2}\right] - \left[1 - 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^2}\right]\right] \\ +y_p\hat{y}\left[\left[1 + 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^2}\right] - \left[1 - 3\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^2}\right]\right] \end{bmatrix} = \frac{kq}{|\vec{r}_p|^3}\begin{bmatrix} x_p\hat{x}\left[6\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^2}\right] \\ +a\hat{x}\left[-2\right] \\ +y_p\hat{y}\left[6\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^2}\right] \end{bmatrix} = \frac{kq}{|\vec{r}_p|^3}\left[-2a\hat{x} + 6\vec{r}_p\left[\frac{\vec{r}_p \bullet \vec{a}}{\vec{r}_p^2}\right]\right]$$

In terms of the dipole moment, we then have:

$$\vec{E}_p \approx \frac{k}{|\vec{r}_p|^3}\left[-\vec{p} + 3\vec{r}_p\left[\frac{\vec{r}_p \bullet \vec{p}}{\vec{r}_p^2}\right]\right] = 3\frac{k\hat{r}_p\left(\hat{r}_p \bullet \vec{p}\right)}{|\vec{r}_p|^3} - \frac{k\vec{p}}{|\vec{r}_p|^3}$$

Where in this case the (physical) electric dipole is $\vec{p} = 2qa\hat{x}$.

(4) Suppose in this case, your dipole had +q at x=a and -q at x=-a. Find an expression for the electric field along *the x-axis* at x>a. You should then be able to show that the electric field behaves as $\mathbf{E_x \approx 4kqa/x^3}$ at distant points along the *x* -axis. Then write the result in terms of the dipole moment.

Here, it's clear that the y-component of the resultant electric field vanishes. It is particularly easy to find the electric field in this case through direct application of the definition of the electric field.

$$\vec{E}_p = \sum_{i=1}^{n} k \frac{q_i}{\left|\vec{r}_p - \vec{r}_i\right|^2} \hat{r}_{ip} = k \frac{q_1}{\left(\vec{r}_p - \vec{r}_1\right)^2} \hat{r}_{1p} + k \frac{q_2}{\left(\vec{r}_p - \vec{r}_2\right)^2} \hat{r}_{2p}$$

We need to get each of the vectors. Also, let's assume for simplicity that we're along the +x axis here. Thus, at a point x_p along the +x axis, we have:

$\hat{r}_{1p} = +\hat{x} = \hat{r}_{2p}$

and $\vec{r}_p = +x_p\hat{x},\ \vec{r}_1 = +a\hat{x}, \vec{r}_2 = -a\hat{x}$. Let's find the distances:

$$\left|\vec{r}_p - \vec{r}_1\right| = \left|x_p - a\right| \Rightarrow \left|\vec{r}_p - \vec{r}_1\right|^2 = \left(x_p - a\right)^2$$
$$\left|\vec{r}_p - \vec{r}_2\right| = \left|x_p + a\right| \Rightarrow \left|\vec{r}_p - \vec{r}_2\right|^2 = \left(x_p + a\right)^2$$

so we then have:

$$E_p = k\frac{+q}{\left(x_p - a\right)^2} + k\frac{-q}{\left(x_p + a\right)^2} = kq\left(\frac{1}{x_p^2 - 2ax_p + a^2} - \frac{1}{x_p^2 + 2ax_p + a^2}\right) =$$
$$= kq\left(\frac{x_p^2 + 2ax_p + a^2 - x_p^2 + 2ax_p - a^2}{\left(x_p^2 + a^2\right)^2 - 4a^2x_p^2}\right) = kq\left(\frac{4ax_p}{x_p^4 - 2a^2x_p^2 + a^4}\right) = 4kqa\left(\frac{x_p}{\left(x_p^2 - a^2\right)^2}\right)$$

As x_p gets large, the only really important term of the denominator is x_p. Thus:

$\vec{E} \approx \frac{4kqa}{\left|x_p\right|^3}\hat{x}$ in the +x region of space. In the –x region of space, the electric field is given by

$\vec{E} \approx -\frac{4kqa}{\left|x_p\right|^3}\hat{x}$ In terms of the electric dipole defined above, we then have at large distances the electric field is given by (along the +x-axis):

$$\vec{E}_p \approx 2k\frac{\vec{p}}{x_p^3}$$

If you use our general result, you should obtain the same (approximate) result:

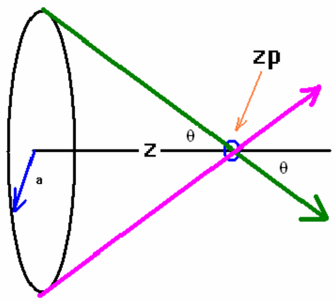
$$\vec{E}_p \approx \frac{k}{\left|\vec{r}_p\right|^3}\left[-\vec{p} + 3\vec{r}_p\left[\frac{\vec{r}_p \bullet \vec{p}}{\vec{r}_p^2}\right]\right] = 3\frac{k\hat{r}_p\left(\hat{r}_p \bullet \vec{p}\right)}{\left|\vec{r}_p\right|^3} - \frac{k\vec{p}}{\left|\vec{r}_p\right|^3} = 3\frac{k\vec{p}}{\left|\vec{r}_p\right|^3} - \frac{k\vec{p}}{\left|\vec{r}_p\right|^3} = 2\frac{k\vec{p}}{\left|\vec{r}_p\right|^3} = 2\frac{k\vec{p}}{x_p^3}$$

Remember, however, our expression for the dipole:

$$\vec{E}_p \approx \frac{k}{\left|\vec{r}_p\right|^3}\left[-\vec{p} + 3\vec{r}_p\left[\frac{\vec{r}_p \bullet \vec{p}}{\vec{r}_p^2}\right]\right] = 3\frac{k\hat{r}_p\left(\hat{r}_p \bullet \vec{p}\right)}{\left|\vec{r}_p\right|^3} - \frac{k\vec{p}}{\left|\vec{r}_p\right|^3}$$

is really only valid for r>>a whereas doing the exact calculation is always valid (since it is without approximation). This means that you can not always start with the field for the dipole to represent any dipole you run into! However at those times when you are in the correct region for approximation, it is appropriate to use this result.

(5) Suppose that you have a ring of radius r=a and total charge Q located in the x-y plane. What is the electric field for points along the symmetry axis of this ring? How does this field behave along the axis at distant points *along the symmetry axis*?



Non-calculus version

This picture showing this particular situation is to the left. The symmetry of the problem allows me to say that the only components of the electric field which survive will lie along the z-axis (i.e. the off-axis components of the electric field cancel) at points along the z-axis.

In this case, then, we have $\vec{E}_{total} = \sum_j \vec{E}_j = \sum_j |\vec{E}_j| \hat{r}_{jp}$

The angle θ is the same no matter where on the ring you look from the symmetry axis (at a fixed z_p). Also, the distance from the ring to the point z_p is the same for every point along the ring. To determine the electric field, write the charge on the ring in terms of the *charge density* on the ring. If you consider that the ring has a total length given by: $2\pi a$, the total charge Q on the ring is given by: $Q = (2\pi a)\lambda$ where I am representing a linear charge density here by λ. The electric field from a very small section at the top of the ring is given by:

$$\vec{E}_{j_+} = \frac{kq_j}{\left[\sqrt{a^2+z_p^2}\right]^2} \frac{z_p\hat{z}-a\hat{y}}{\sqrt{z_p^2+a^2}}$$

where the subscript "+" means I've picked the point from the top of the ring. The electric field coming from a point on the bottom of the ring (exactly opposite from the previous position) is given by:

$$\vec{E}_{j_-} = \frac{kq_j}{\left[\sqrt{a^2+z_p^2}\right]^2} \frac{z_p\hat{z}+a\hat{y}}{\sqrt{z_p^2+a^2}}$$

If I add these two electric fields, I get the result:

$$\vec{E}_{j_+ + j_-} = \frac{k(2q_j)}{\left[\sqrt{a^2+z_p^2}\right]^2} \frac{z_p}{\sqrt{z_p^2+a^2}}\hat{z}$$

Now you need to determine how many such charge pairs there are on the ring. If you let the small charge q_j be represented by $q_j = \lambda(a(\Delta\varphi))$ where $\Delta\varphi$ represents a small angle, then we can rewrite the electric field as:

$$\vec{E}_{j_+ + j_-} = \frac{k(2\lambda a(\Delta\varphi))}{\left[\sqrt{a^2+z_p^2}\right]^2} \frac{z_p}{\sqrt{z_p^2+a^2}}\hat{z}$$

If we now let $\Delta\varphi$ represent ½ of the total angle of the ring (remember, I'm adding up charge pairs here), the electric field from the entire ring becomes:

$$\vec{E}_p = \frac{k(2\lambda a\pi)}{\left[\sqrt{a^2+z_p^2}\right]^2} \frac{z_p}{\sqrt{z_p^2+a^2}}\hat{z}$$

In terms of the total charge Q placed upon the ring, we thus have:

$$\vec{E}_p = \frac{kQ}{\left[\sqrt{a^2+z_p^2}\right]^2} \frac{z_p}{\sqrt{z_p^2+a^2}}\hat{z} = \frac{kQz_p}{\left[a^2+z_p^2\right]^{3/2}}\hat{z}$$

Let's also look at how this behaves as x gets large.

$$\left(a^2+z_p^2\right)^{3/2} = \left(z_p^2\right)^{3/2}\left(1+\frac{a^2}{z_p^2}\right)^{3/2} = z_p^3\left(1+\frac{3}{2}\frac{a^2}{z_p^2}+\ldots\right) \approx z_p^3$$

At large distances, $\vec{E} \approx \frac{kQ}{z_p^2}\hat{z}$ (the ring looks a lot like a point charge).

Calculus version

For calculus people, this is your first example of how to integrate over a continuous charge distribution. I'll do it quite directly without looking at symmetry.

The differential charge density is given by:

$$dq = \lambda\, a\, d\varphi$$

with the charge density defined as previously. You can verify that integrating over this charge density gives you the total charge if $\lambda = \frac{Q}{2\pi a}$

We find the electric field by:

$$\vec{E}_p = \int_{\substack{\text{all}\\ \text{charges}}} d\vec{E}$$

Now the magnitude of the electric field arising from dq is given by:

$$\left|d\vec{E}\right| = k\frac{dq}{\left[z_p^2+a^2\right]}$$

The vector pointing towards a charge location is given by:

$$\vec{r}_i = x_i\hat{x} + y_i\hat{y} = a\cos(\varphi)\,\hat{x} + a\sin(\varphi)\,\hat{y}$$

Note: How did I get this? Look at the picture and remember that when you convert from Cartesian coordinates to polar coordinates, the transformation is:

$$x = a\cos\varphi \;\;\&\;\; y = a\sin\varphi$$

where I am using φ to represent the polar angle in the x-y plane.

It is very important also to notice that I kept the unit vectors in Cartesian coordinates. This is a rule you do not want to break when integrating over unit vectors since it is only the Cartesian unit vectors which are constant in space!

The vectors that we need are given by:

$$\vec{r}_{ip} = \vec{r}_p - \vec{r}_i = -a\cos(\varphi)\,\hat{x} - a\sin(\varphi)\,\hat{y} + z_p\hat{z}$$

$$\hat{r}_{ip} = \frac{-a\cos(\varphi)\hat{x} - a\sin(\varphi)\hat{y} + z_p\hat{z}}{\sqrt{z_p^2+a^2}}$$

I am now ready to calculate the integral:

$$\vec{E}_p = k\int_{\varphi=0}^{\varphi=2\pi}\left\{\frac{-a\cos(\varphi)\hat{x} - a\sin(\varphi)\hat{y} + z_p\hat{z}}{\left[a^2+z_p^2\right]^{3/2}}\right\}(\lambda\, a\, d\varphi)$$

This integral can be written as 3 integrals:

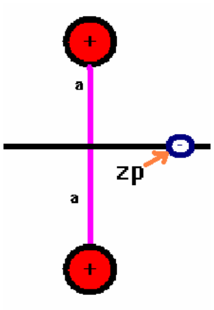
$$\vec{E}_p = k\frac{a\lambda}{\left[a^2+z_p^2\right]^{3/2}}\left\{-a\hat{x}\int_{\varphi=0}^{\varphi=2\pi}\cos(\varphi)\,d\varphi - a\hat{y}\int_{\varphi=0}^{\varphi=2\pi}\sin(\varphi)\,d\varphi + z_p\hat{z}\int_{\varphi=0}^{\varphi=2\pi}d\varphi\right\}$$

The first two integrals vanish, giving us the result:

$$\vec{E}_p = k\frac{(2\pi)\lambda a z_p}{\left[z_p^2+a^2\right]^{3/2}}\hat{z} = k\frac{Q z_p}{\left[z_p^2+a^2\right]^{3/2}}\hat{z}$$

Do study the steps that I've used to do this problem. If you follow these steps in this way, the problem of integration over continuous charge distributions will be straight-forward.

Here is a nice application of what you have learned that also ties some things together!



Suppose you have a crystal which has two positive charges located as shown and an electron is located along the symmetry axis between the two charges at a distance z_p from the center which is very small compared to a. Let's see what happens.
This problem is unlike the dipole problem in that each of the charges is the same. However, looking at the non-calculus approach to the ring problem (problem 5), it is immediately apparent what the electric field is along the symmetry axis. The electric field is given by:

$$\vec{E}_{j_+ + j_-} = \frac{k(2q_j)}{\left[\sqrt{a^2+z_p^2}\right]^2}\frac{z_p}{\sqrt{z_p^2+a^2}}\hat{z} = \frac{2kq\,z_p}{\left[a^2+z_p^2\right]^{3/2}}\hat{z}$$

Now we're going to look at this expression in the limit that $|z_p| << a$. We again use the binomial expansion but we need to rewrite the denominator slightly.

$$\left[a^2+z_p^2\right]^{3/2} = a^3\left[1+\left(\frac{z_p}{a}\right)^2\right]^{3/2} \approx a^3\left(1+\frac{3}{2}\left(\frac{z_p}{a}\right)^2+\ldots\right)$$

The leading term is then a^3 which gives us the approximate electric field at the center as:

$$\vec{E}_p \approx \frac{2kqz_p}{a^3}\hat{z}$$

Now let's find the electrostatic force on the electron which is trapped in such a situation. This is easily seen to be given by:

$$\vec{F} = \left(q_{electron}\right)\vec{E}_p = -e\vec{E}_p = -\frac{2keq}{a^3}z_p\hat{z}$$

This force is linear in the displacement variable and restoring. If you compare this force to the Hooke's law force ($\vec{F} = -Kx\hat{x}$) then you would expect to see the electron oscillate with an angular frequency and thus a frequency given by:

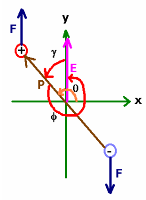
$$\omega = \sqrt{\frac{K}{m_e}} = \sqrt{\frac{2keq}{a^3m_e}} \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{2keq}{a^3m_e}}$$

You often hear that molecules act like springs connected to masses but this really shows the effect. The electron will oscillate (and thus, it will store energy). The problem is that this is a classical calculation. It is, however, very easy at this point, with a little bit of quantum mechanics to obtain an energy spectrum for the electron trapped between two positively charged ions like this! Look at (for further information):
http://www.lyon.edu/webdata/users/shutton/phy335-fall2003/QSHO.doc

The energy spectrum will be given by: $E_n = \hbar\omega\left(n+\frac{1}{2}\right); n = 1,2,3,\ldots; \omega = \sqrt{\frac{K}{m_e}}$.

Here, you also see Planck's constant which is given by: $\hbar = \frac{h}{2\pi} = 1.0546\times10^{-34}\,J\bullet s$

Note: you won't be tested on this bit in the box.



Here is another nice application related to the electric dipole. Suppose that we apply a uniform electric field along the y-axis of the dipole in problem 4. The external electric field is given by:

$$\vec{E} = E\hat{y}$$

The angle between $\vec{p}$ and $\vec{E}$ is φ. The angle between $\vec{E}$ and $\vec{p}$ is γ. These two angles are related by: $\varphi + \gamma = 360^0$. The angle between the positive x-axis and P is θ.

The coordinates of the charges are:

$$\vec{r}_+ = a\cos(\theta)\hat{x} + a\sin(\theta)\hat{y} \;\; : \;\; \vec{r}_- = a\cos(\theta + 180^0)\hat{x} + a\sin(\theta + 180^0)\hat{y}$$

The torque on the positive charge is given by:

$$\vec{\tau}_+ = \vec{r}_+ \times \vec{F}_+ = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a\cos(\theta) & a\sin(\theta) & 0 \\ 0 & E|q| & 0 \end{vmatrix} = \hat{x}(0) - \hat{y}(0) + \hat{z}\left(Ea|q|\cos(\theta)\right) = Ea|q|\cos(\theta)\hat{z}$$

Since $\cos(\beta) = -\cos(\beta + 180^0)$, and the fact that the negative charge has a negative sign, the torque from the negative charge is the same as for the positive charge.

$$\vec{\tau}_- = \vec{\tau}_+ \Rightarrow \vec{\Gamma} = -2Ea|q|\cos(\theta)\hat{z} = 2Ea|q|\cos(\theta) = |P||E|\cos(\theta)$$

Now, there is also another connection:

$$\gamma + 90^0 = \theta \Rightarrow \gamma = \theta - 90^0$$

$$360^0 - \gamma = \varphi \Rightarrow 360^0 - \theta + 90^0 = 450^0 - \theta = \varphi \Rightarrow \theta = 450^0 - \varphi$$

We thus have the net torque on the dipole given as:

$$\vec{\Gamma} = |\vec{p}||\vec{E}|\cos(450^0 - \varphi)\hat{z} = |\vec{p}||\vec{E}|\left\{\cos(450^0)\cos(\varphi) + \sin(450^0)\sin(\varphi)\right\}\hat{z}$$

$$\Rightarrow \vec{\Gamma} = |\vec{p}||\vec{E}|\sin(\varphi)\hat{z} = \vec{p} \times \vec{E}$$

where the angle is measured starting with the positive $\vec{p}$ axis and rotating around in the positive manner (counterclockwise). On the other hand, if you want to relate this to the angle γ which starts along the Positive E direction and rotates counterclockwise towards p, then you have

$$\sin(\varphi) = \sin(360^0 - \gamma) = \sin(360^0)\cos(\gamma) - \sin(\gamma)\cos(360^0) = -\sin(\gamma)$$

Thus the torque is given by:

$$\vec{\Gamma} = -|\vec{p}||\vec{E}|\sin(\gamma)\hat{z} = \vec{E} \times \vec{p} = \vec{p} \times \vec{E}$$

Now here is why I worry so much about the sign of this torque: if the sign is wrong, simple harmonic oscillation won't result from the analysis below. In particular, you want to fix yourself onto the electric field vector and watch the dipole oscillate about you, rather than fixing yourself on the dipole and watching the electric field oscillate.
According to Newton's laws, we have that a torque produces an angular acceleration:

$$\Gamma = I\alpha$$

So the equation of motion is given by:

$$-|\vec{p}||\vec{E}|\sin(\gamma) = I\alpha \Rightarrow \alpha + \tfrac{|\vec{p}||\vec{E}|}{I}\sin(\gamma) = 0$$

Calculus students write this as:

$$\frac{d^2\gamma}{dt^2}+\frac{|\vec{p}||\vec{E}|}{I}\sin(\gamma)=0$$

Now if you consider only small angles, then:

$$\sin(\gamma)\approx\gamma$$

This means that simple harmonic oscillation will result with a frequency of oscillation given by:

$$\omega=\sqrt{\frac{|\vec{p}||\vec{E}|}{I}}=\sqrt{\frac{\vec{p}E}{2ma^2}}=\sqrt{\frac{2qaE}{2ma^2}}=\sqrt{\frac{qE}{ma}}=2\pi f\Rightarrow f=\frac{1}{2\pi}\sqrt{\frac{qE}{ma}}$$

As before, the energy spectrum of the oscillating dipole would be quantized and thus:

$$E_n=\hbar\omega\left(n+\tfrac{1}{2}\right);n=1,2,3,...;\omega=\sqrt{\frac{qE}{ma}}$$

Here, you also see Planck's constant which is given by: $\hbar=\frac{h}{2\pi}=1.0546\times10^{-34}\,\text{J}\bullet\text{s}$

This is yet one more example of where concepts from the first semester are very important in the second semester of physics for a more complete picture.

Incidentally, you'll also need to know something about the electric polarization. The electric polarization of a material $\vec{P}$ is defined as the dipole moment per unit volume of the material. This can be difficult to calculate but it is a vector quantity.

You can also calculate the work required to orient a dipole from some angle θ (as I have defined it above) to the x-axis to some angle (where θ=0).

$$W=\int_{\theta}^{\theta=0}\Gamma d\theta=\int_{\theta}^{\theta=0}pE\cos(\theta)d\theta=pE\sin(\theta)\Big|_{\theta}^{0}=-pE\sin(\theta)$$

Since

$$\gamma=\theta-90^0\Rightarrow\theta=\gamma+90^0\Rightarrow\sin(\theta)=\sin\left(\gamma+90^0\right)=\sin\gamma\cos\left(90^0\right)+\sin\left(90^0\right)\cos\gamma=\cos\gamma$$

we can rewrite this result in terms of the dot product. Thus, in terms of the angle between $\vec{E}$ and $\vec{p}$, we have:

$$U=-\vec{p}\bullet\vec{E}$$

Which would correspond to the energy of a dipole in an external electric field. This is important classically for a lot of dipoles in an external electric field. You can do an average over angles using Boltzman statistics to obtain an average angle (this leads to an equation known as the Langevin equation).