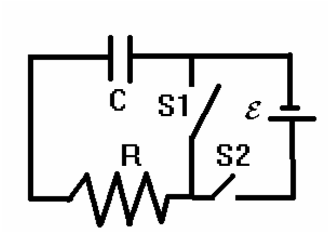
	Physics 250: Worksheet 7	Name: _______________

(1) A resistor R_1=10Ω is connected in series with a resistor R_2=100Ω. A current I=0.1 A is present through the circuit. What is the power radiated in each resistor and also in the total circuit?

(2) A battery is measured to provide an emf of 1.5 V. When a 10Ω resistor is placed across the battery, a current of 0.125A is observed to flow. What is the internal resistance of the battery?

(3) A resistor and a capacitor are connected in series to a battery as shown. If C=1μf, R=1000 kΩ and the battery can supply an emf of 10 V, answer the following questions.



(a) If switch S1 remains open while S2 is closed at t=0, after an infinite amount of time, what will the potential difference across the capacitor be?

(b) How long will it take until the capacitor has a potential difference of 5 V?

(c) After the capacitor is charged to its maximum value, Switch S2 is opened while switch S1 is closed so that the capacitor discharges through the resistor only. How long will it take for the voltage across the capacitor to drop by a factor of ½?

(d) What is the maximum charge which is possible on the capacitor?

(e) What is the maximum current through the discharging circuit?

Power in resistances

The instantaneous power radiated by a resistor is given by $P = I^2R$

Proof: Looking at the definition of electrical potential and potential difference, we have:

$$U = QV$$

where U is the work required to move charge Q across the potential difference V.

Now however, Q has a time dependence, resulting in a current I. Thus, if V is constant,

$$P \equiv \frac{\Delta U}{\Delta t} = V\frac{\Delta Q}{\Delta t} = IV$$

We can write this in one of 3 ways using Ohm's law:

$$P = IV = I^2R = \frac{V^2}{R}$$

Calculus students would like to see the second line written as:

$$P \equiv \frac{dU}{dt} = V\frac{dQ}{dt} = IV$$

As a passing reminder, in the SI system, the units of power are Watts (1W=1J/1s).

What about the emf from a battery and the like?

It's important to know how batteries and voltage sources in general work. The interesting thing is that most people use batteries but either don't know how they work or believe incorrect things about how they work. A battery provides a potential difference internally due to chemical processes. This is called the **emf** (**$\mathcal{E}$**) of the battery.

A battery also has internal resistance which serves to decrease the actual potential which is ultimately supplied to a device from the terminals of the battery **if the battery is connected to a circuit in which current is flowing.** We'll call the potential difference that the terminals of the battery provide the "terminal voltage."

I've tried to emphasize this last point. You might ask … what is the effect of the internal resistance on the terminal voltage? The answer to this depends upon whether the battery is connected to a circuit or not. As it turn out, both of these cases are related.

Case I:

Suppose a battery with an emf $\mathcal{E}$ has an internal resistance R_i and is connected to a circuit which has a resistance R. A current I is flowing (from the battery) into the circuit. The terminal voltage is given by:

$$V_{terminal} = \mathcal{E} - IR_i$$

This is somewhat unpleasing since you've got to find I in order to determine the terminal voltage. We can, however do this:

$$0 = \mathcal{E} - IR_i - IR \Rightarrow I = \frac{\mathcal{E}}{(R_i + R)}$$

We can then say that the terminal voltage would be given by:

$$V_{terminal} = \mathcal{E} - \frac{\mathcal{E}}{(R_i + R)} R_i = \mathcal{E}\left(1 - \frac{R_i}{R_i + R}\right) = \mathcal{E}\left(\frac{R}{(R_i + R)}\right)$$

Every reference I've seen talks about the internal resistance as being a series resistance. In fact, it might be more appropriate to refer to it as a parallel resistance. In this case, things are somewhat more complicated. However, since batteries do generally produce a potential difference for long periods of time (which would not necessarily be true if the internal resistance were parallel), I believe it is safe to assume here that it will act as a series resistor and not a parallel resistor.

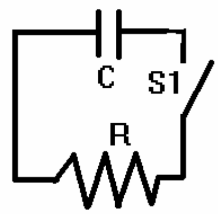
Case II:

Suppose a battery with an emf $\mathcal{E}$ has an internal resistance R_i and is connected to a circuit which has an infinite resistance (and thus no current flows from the battery into the circuit). Look at the last expression above. It is very clear that in this case, the terminal voltage and the emf would be the same. Thus, since an infinite resistance corresponds to an open circuit, this is what you would measure if you placed a good voltmeter across the terminals of a battery.

Time dependence of an RC circuit:

Here is a note on capacitors: strictly said, it is not proper to refer to "charging" a capacitor. In fact, what is happening is that the charge is being separated by the establishment of an emf that forces the charge to separate across the plates of the capacitor. At the present time, the single mechanism we have that will allow this to happen is if the charge travels through wires and not between the capacitor plates. Thus when I say "charging" or "discharging" (and Q), I am referring to the amount of charge separated. In this sense, if I separate 1 positive charge from 1 negative charge, that would be a charge of e^- (not $2e^-$).

I: Discharging capacitances



Consider this circuit. Initially the capacitor is charged with Q. At t=0 switch S1 is closed. How do Q, I and V behave in time?

The sum of the potential increases and drops on a closed circuit is zero. You'll later know this as one of Kirchoff's laws, but for now you can see it also is a consequence of energy conservation. This means that for the circuit shown above:

$$V_c + V_R = 0$$

Now strictly said, this equation ignores any effects due to the finite speed of propagation of the signal through the circuit. It gets more complicated if you need to consider this effect. We can rewrite this using Ohm's law and also the definition of capacitance as:

$$V_c = \frac{Q}{C} \text{ and } V_R = IR \;\Rightarrow \frac{Q}{C} + IR = 0 \Rightarrow Q + I(RC) = 0$$

We now use the definition of current to establish the time dependence of the circuit. It is important to understand just where this current I comes from. Here, like the case of connecting a battery to the circuit, the current comes from charges that are moving through the circuit. The connection between this current and the rate of change of charge separation within the capacitor is straight-forward:

$$\text{Non-calculus: } I = \frac{\Delta Q}{\Delta t} \quad \text{Calculus: } I = \frac{dQ}{dt}.$$

However, to fully appreciate this, you need to realize that the total charge separation is decreasing with time. This means then that if Q refers to the positive charge separation, the rate of this charge separation changing is **negative**. For our purposes here, this forces a negative sign to appear in our calculations which follow.

More details about current

There is another way to realize this: in fact, I is closely related to a vector quantity. Let me show you the connection in a different light. Suppose on a region of space, characterized by a resistance R you have a potential difference which varies as $V = ax$. The electric field in this region of space is given by:

non-calculus: $\vec{E} = -\frac{\Delta V}{\Delta x}\hat{x} = -a\hat{x}$ Calculus: $\vec{E} = -\vec{\nabla}V = -a\hat{x}$

Now let's look at how Ohm's law becomes involved: if $V = IR$, we can't necessarily say that I varies. In fact, the connection would become: $\vec{J} = \sigma\vec{E}$. Here, $\vec{J}$ is a "current density" and has units of Amps/m^2. σ is called the "conductivity" (with units of $\frac{1}{\Omega m} \equiv \text{mho}$) of the material and is related to the resistivity in DC circuits by: $\sigma = \frac{1}{\rho}$.The current is obtained from the current density by adding up the flux of current through the face of an area A normal to the current so that:

non-calculus: $I = \vec{J} \bullet \vec{A}$ Calculus: $I = \oint \vec{J} \bullet d\vec{A}$

We will need more of this soon when we talk about magnetism.

Now back to our equation which was: $Q + I(RC) = 0$

non-calculus:

$$I = \frac{\Delta Q}{\Delta t} \Rightarrow Q = -\frac{\Delta Q}{\Delta t}(RC) \Rightarrow \frac{\Delta Q}{Q} = -\frac{\Delta t}{RC}$$

you can recognize that this is going to represent a logarithmic behavior. Thus:

$$\frac{\Delta Q}{Q} = -\frac{\Delta t}{RC} \Rightarrow \ln\left(\frac{Q}{Q_0}\right) = -\frac{t}{RC}$$

calculus:

$$Q + \frac{dQ}{dt}(RC) = 0 \Rightarrow Q = -(RC)\frac{dQ}{dt} \Rightarrow \frac{dQ}{Q} = -\frac{dt}{RC} \Rightarrow \ln\left(\frac{Q}{Q_0}\right) = -\frac{t}{RC}$$

Take the exponential of both sides to obtain:

$$Q = Q_0 e^{-\frac{t}{RC}}$$

We often represent RC by the symbol τ and it is called the "time constant" of the circuit:

$$Q = Q_0 e^{-\frac{t}{\tau}}$$

The voltage dependence is easy to obtain using the definition of capacitance:

$$V = V_0 e^{-\frac{t}{\tau}}$$

We can obtain the dependence of the current by looking at the rate of change of charge.

Non-calculus:

$$Q + \tau\frac{\Delta Q}{\Delta t} = 0 \Rightarrow Q + \tau I = 0 \Rightarrow \frac{\Delta Q}{\Delta t} = -\tau\frac{\Delta I}{\Delta t} \Rightarrow I = -\tau\frac{\Delta I}{\Delta t} \Rightarrow \frac{\Delta I}{I} = -\frac{\Delta t}{\tau}$$

Calculus:

$$Q + \tau\frac{dQ}{dt} = 0 \Rightarrow Q + \tau I = 0 \Rightarrow \frac{dQ}{dt} = -\tau\frac{dI}{dt} \Rightarrow I = -\tau\frac{dI}{dt} \Rightarrow \frac{dI}{I} = -\frac{dt}{\tau}$$

Again you can recognize that this is a logarithmic dependence:

$$\ln\left(\frac{I}{I_0}\right) = -\frac{t}{\tau} \Rightarrow I = I_0 e^{-\frac{t}{\tau}}$$

The 3 results we thus have are these:

$$Q = Q_0 e^{-\frac{t}{\tau}} : V = V_0 e^{-\frac{t}{\tau}} : I = I_0 e^{-\frac{t}{\tau}}$$

you can also find out how the energy radiated decays with time:

$$P = IV = I_0 V_0 e^{-2\frac{t}{\tau}} = P_0 e^{-2\frac{t}{\tau}}$$

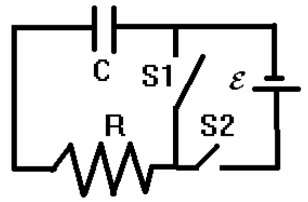
The instantaneous energy stored in the capacitor is given by:

$$U = \frac{1}{2}CV^2 = \frac{1}{2}CV_0^2 e^{-2\frac{t}{\tau}} = U_0 e^{-2\frac{t}{\tau}}$$

We need to express the initial quantities in terms that we can understand. If a battery producing an emf $\mathcal{E}$ was used to charge the capacitor, then we would have the following results:

$$V_0 = \mathcal{E} : I_0 = (-)\frac{V_0}{R} = \frac{\mathcal{E}}{R} = C\frac{\mathcal{E}}{\tau} : Q_0 = \frac{V_0}{C} = \frac{\mathcal{E}}{C} = R\frac{\mathcal{E}}{\tau} : P_0 = \frac{\mathcal{E}^2}{R} : U_0 = \frac{1}{2}C\mathcal{E}^2$$

II: Charging capacitances



Consider this circuit. It's pretty much the same circuit as I had earlier except that now I've added a battery to it. In fact, to get exactly the same operation as earlier, with the capacitor charged, close switch S1 while leaving switch S2 open. Let me show you how you can charge the capacitor with this circuit. Close switch S2 while leaving switch S1 open and the capacitor will charge up to the point where the emf from the battery is equal to the potential difference across the capacitor.

This is the simplest way in the world (I think) to see this effect.

Point (1): the highest voltage that the capacitor will achieve is equal to the emf of the battery that is charging the capacitor. This is because the capacitor ultimately achieves a potential which is equal and opposite to the potential which is charging the capacitor at which point current stops flowing through the circuit. If charge did not unseparate through the circuit, the capacitor would reach this potential difference instantly.

Point (2): as the capacitor is charging, it is also trying to discharge the whole time *across the circuit in the opposite direction to that which it is charging*. This gives an exponential decrease to the potential across the capacitor.

Add these two effects:

$$V(t) = V_{max}\left(1 - e^{-\frac{t}{\tau}}\right) = \mathcal{E}\left(1 - e^{-\frac{t}{\tau}}\right)$$

The charge across the capacitor is given by:

$$Q(t) = Q_{max}\left(1 - e^{-\frac{t}{\tau}}\right) = C\mathcal{E}\left(1 - e^{-\frac{t}{\tau}}\right) = \frac{\mathcal{E}\tau}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$

The current through the circuit (not! across the capacitor) is given by:

$$I = I_{max}e^{-\frac{t}{\tau}} = \frac{\mathcal{E}}{R}e^{-\frac{t}{\tau}} = C\frac{\mathcal{E}}{\tau}e^{-\frac{t}{\tau}}$$

(we don't have that pesky problem with minus signs here).

The instantaneous power radiated across the resistor is given by:

$$P = I^2R = R\mathcal{E}^2e^{-\frac{t}{\tau}}$$

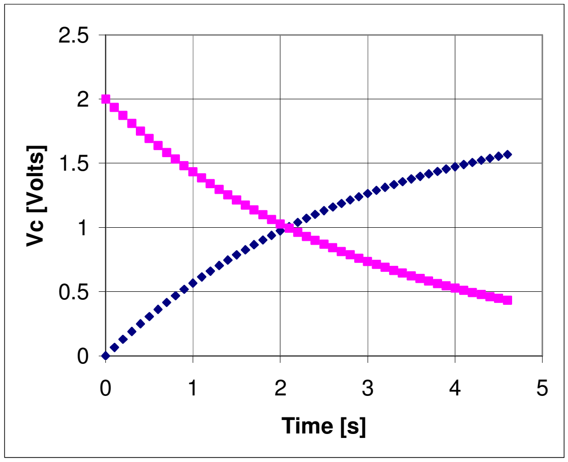
The instantaneous energy stored in the capacitor is given by:

$$U = \frac{1}{2}CV^2 = \frac{1}{2}C\mathcal{E}^2\left(1 - e^{-\frac{t}{\tau}}\right)^2$$

Actually, the capacitor ultimately stores an amount of energy given by $\frac{1}{2}C\mathcal{E}^2$. The battery did an amount of work given by $C\mathcal{E}^2$. Thus exactly ½ of the energy was radiated by the resistor and ½ of the energy was stored in the capacitor. You can prove this last point by adding up the power expended for all times. Technically, the battery will do additional work against the internal resistance of the battery also.

You'll find an interactive spreadsheet below.
Innovations in electronic physics text test. ☺

t	V(charge)	V(discharge)	Vmax	τ
0	0	2	2	3
0	0.065568	1.934432201		
0	0.128986	1.87101397		
0	0.190325	1.809674836		
0	0.249653	1.750346638		
1	0.307037	1.69296345		
1	0.362538	1.637461506		
1	0.416221	1.583779133		
1	0.468143	1.531856677		
1	0.518364	1.481636441		
1	0.566937	1.433062621		
1	0.613919	1.38608124		
1	0.65936	1.340640092		
1	0.703311	1.296688682		
1	0.745822	1.254178171		
2	0.786939	1.213061319		
2	0.826708	1.173292439		
2	0.865173	1.134827338		
2	0.902377	1.097623272		
2	0.938361	1.061638901		
2	0.973166	1.026834238		
2	1.006829	0.993170608		
2	1.039389	0.960610602		
2	1.070882	0.929118041		
2	1.101342	0.898657928		
3	1.130804	0.869196417		
3	1.159299	0.840700769		
3	1.186861	0.813139319		
3	1.213519	0.786481442		
3	1.239302	0.760697513		
3	1.264241	0.735758882		
3	1.288362	0.711637837		
3	1.311692	0.688307574		
3	1.334258	0.665742167		
3	1.356083	0.643916543		
4	1.377194	0.622806448		
4	1.397612	0.602388424		
4	1.41736	0.582639782		
4	1.436461	0.563538578		
4	1.454936	0.545063586		
4	1.472806	0.527194276		
4	1.490089	0.509910792		
4	1.506806	0.493193928		
4	1.522975	0.477025108		
4	1.538614	0.461386365		
5	1.55374	0.44626032		
5	1.56837	0.431630167		



(1) A resistor $R_1=10\Omega$ is connected in series with a resistor $R_2=100\Omega$. A current I=0.1 A is present through the circuit. What is the power radiated in each resistor and also in the total circuit?

Solution:

The instantaneous power radiated in each resistor is given by: $P=I^2R$. Thus, the powers radiated are $P_1=0.1W$ and $P_2=1W$. The total power radiated is 1.1 W. Suppose that the two resistors were connected in parallel and a current of 0.1A was injected into the circuit element. Then, we need to find the equivalent resistance to answer this question. The equivalent resistance is: $R_{eq}=9.091\Omega$. The power radiated through this equivalent resistance is $P=I^2R_{eq}=0.091W$. It's a bit harder to find the power radiated by each resistor. However, since the potential difference across each resistor is the same, we have:

$V = R_1I_1 = R_2I_2 = R_{eq}I \Rightarrow I_1 = \frac{R_{eq}}{R_1}I$ and $I_2 = \frac{R_{eq}}{R_2}I$. In this case, then we have

$$I_1 = \tfrac{9.091}{10}(0.1) = 0.091A \Rightarrow P_1 = 8.26x10^{-2}W$$

$$I_2 = \tfrac{9.091}{100}(0.1) = 0.0091A \Rightarrow P_2 = 8.26x10^{-3}W$$

$$P = P_1 + P_2 = 0.091W$$

(2) A battery is measured to provide an emf of 1.5 V. When a 10Ω resistor is placed across the battery, a current of 0.125A is observed to flow. What is the internal resistance of the battery?

Solution:

The battery provides an emf and the resistance are assumed to be in series. Thus:

$$\mathcal{E} = I(R_{int} + R_{ext})$$

It's pretty straight forward now to get the internal resistance:

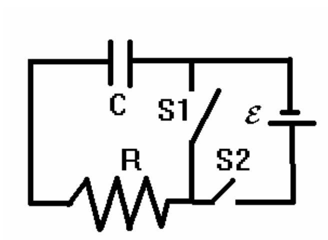
$$\tfrac{\mathcal{E}}{I} - R_{ext} = R_{int}$$

In the present problem, we then have:

$$\tfrac{1.5}{0.125} - 10 = 2\Omega = R_{int}$$

How do you measure emf? The answer is to use a good voltmeter and measure the potential across the terminals of a battery. You will do this in lab.

(3) A resistor and a capacitor are connected in series to a battery as shown. If C=1μf, R=1000 kΩ and the battery can supply an emf of 10 V, answer the following questions.



(a) If switch S1 remains open while S2 is closed at t=0, after an infinite amount of time, what will the potential difference across the capacitor be?

(b) How long will it take until the capacitor has a potential difference of 5 V?

(c) After the capacitor is charged to its maximum value, Switch S2 is opened while switch S1 is closed so that the capacitor discharges through the resistor only. How long will it take for the voltage across the capacitor to drop by a factor of ½?

(d) What is the maximum charge which is possible on the capacitor?

(e) What is the maximum current through the discharging circuit?

(a) The maximum potential difference across the capacitor will be equal to the emf of the battery which is 10V.

(b) The time constant of the circuit is given by:

$$\tau = RC = \left(1x10^{6}\right)x\left(1x10^{-6}\right) = 1s$$

The equation describing how the potential difference across the capacitor increases is:

$$V(t) = V_{max}\left(1 - e^{-\frac{t}{\tau}}\right) = \mathcal{E}\left(1 - e^{-\frac{t}{\tau}}\right)$$

We need to solve this equation:

$$\frac{V}{V_{max}} = 1 - e^{-\frac{t}{\tau}} \Rightarrow 1 - \frac{V}{V_{max}} = e^{-\frac{t}{\tau}} \Rightarrow -\frac{t}{\tau} = \ln\left(1 - \frac{V}{V_{max}}\right) \Rightarrow t = -\tau\ln\left(1 - \frac{V}{V_{max}}\right)$$

Thus, the time for any potential difference increase is given by:

$$t = -\tau\ln\left(1 - \frac{V}{V_{max}}\right)$$

For a change of ½, then:

$$t = -\tau\ln\left(1 - \frac{V}{V_{max}}\right) = -(1)\ln\left(1 - \frac{\left(\frac{1}{2}V_{max}\right)}{V\,max}\right) = -\ln(1 - 0.5) = -\ln(0.5) = 0.693s$$

(c) The equation describing how the potential difference across the capacitor decrease is:

$$V(t) = V_{max}e^{-\frac{t}{\tau}}$$

We want to find out how long it takes for the initial voltage to drop by a factor of ½. Thus:

$$\frac{V}{V_{max}} = e^{-\frac{t}{\tau}} \Rightarrow t = -\tau\ln\left(\frac{V}{V_{max}}\right) = -\tau\ln\left(\frac{1}{2}\right) = 0.693s$$

The time to drop by a factor of $\frac{1}{2}$ of $\frac{1}{2}$ so that $\frac{V}{V_{max}} = \frac{1}{4}$ is 0.693+0.693=1.386s.

The time to drop by a factor of $\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{2}$ so that $\frac{V}{V_{max}} = \frac{1}{8}$ is 3(0.693s)=2.079s.

This is the way that exponential decay works. Each succeeding halving of the population takes the same amount of time as the previous halving of the population. Exponential change will happen when the change in a population is proportional to the population.

An aside to help you not go bankrupt (you will not see this on one of my tests)

Before you borrow money, be aware of exactly how exponential growth works since the amount of money you owe will depend upon not only how much you borrow but also the amount of time required to pay it back.

Suppose you borrow money at a 7% rate, compounded instantly. How long will it be until you owe twice the money you initially borrowed, assuming you pay nothing on the loan?

Answer: $\$ = \$_0 e^{0.07t} \Rightarrow \frac{1}{0.07}\ln\left(\frac{\$}{\$_0}\right) = t \Rightarrow t = \frac{1}{0.07}\ln\left(\frac{2}{1}\right) = 9.9\text{ years}$

What if you had a credit card debt where the rate was 19%. How long would it take then?

$$t = \frac{1}{0.19}\ln(2) = 3.65\text{ years}$$

Ultimately there's one moral to this story: understanding exponential growth not only is good for your physics grade, but it is extremely important for your financial health!

(d) The maximum charge on a capacitor is given by the limiting behavior of the capacitor when $V=V_{max}$. Since

$$C = \frac{Q}{V} \Rightarrow Q_{max} = CV_{max}$$

In this case, $Q_{max}=10\mu C$.

(e) The maximum current through the discharging circuit will happen at t=0 and will steadily drop to eventually become zero. The instantaneous current is given by:

$$I = I_{max}e^{-\frac{t}{\tau}} = \frac{\mathcal{E}}{R}e^{-\frac{t}{\tau}} = C\frac{\mathcal{E}}{\tau}e^{-\frac{t}{\tau}}$$

Here, the maximum current is then $I_{max} = \frac{\mathcal{E}}{R} = \frac{10V}{1x10^6\,\Omega} = 1x10^{-5}A$