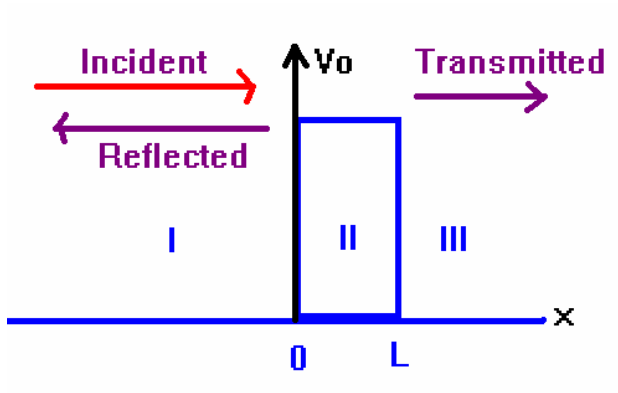
Barriers and Tunneling
Potential Barrier with $E>V_0$



$$k_I = k_{III} = \frac{\sqrt{2mE}}{\hbar} \text{ where V=0}$$

$$k_{II} = \frac{\sqrt{2m(E-V_0)}}{\hbar} \text{ where } V=V_0$$

The 1DTISWE:

$$\frac{d^2\varphi_I}{dx^2} + \frac{2m}{\hbar^2}\varphi_I = 0 \Rightarrow \varphi_I = Ae^{ik_I x} + Be^{-ik_I x}$$

$$\frac{d^2\varphi_{II}}{dx^2} + \frac{2m}{\hbar^2}[E-V_0]\varphi_{II} = 0 \Rightarrow \varphi_{II} = Ce^{ik_{II} x} + De^{-ik_{II} x}$$

$$\frac{d^2\varphi_{III}}{dx^2} + \frac{2m}{\hbar^2}\varphi_{III} = 0 \Rightarrow \varphi_{III} = Fe^{ik_I x} + Ge^{-ik_I x}$$

Assume that the particles are coming from the left moving along the +x direction. In this case, then A represents the incident particles and B represents the reflected particles. In region III, no particles are initially moving in from +infinity, thus G=0.

We thus have:

$$\text{incident}: \varphi_I = Ae^{ik_I x}$$

$$\text{reflected}: \varphi_I = Be^{-ik_I x}$$

$$\text{transmitted}: \varphi_{III} = Fe^{ik_I x}$$

The probability of reflection is given by:

$$R = \frac{|\varphi_I(\text{reflected})|^2}{|\varphi_I(\text{incident})|^2} = \frac{B^*B}{A^*A}$$

The probability of transmission is given by:

$$T = \frac{|\varphi(\text{transmitted})|^2}{|\varphi(\text{incident})|^2} = \frac{F^*F}{A^*A}$$

The particles have to be either transmitted or reflected. This requires:

$$R + T = 1$$

Now to find the particular values of R and T, you need to satisfy the boundary conditions:

$$\text{at } x=0, \varphi_I = \varphi_{II} \text{ and } \frac{\partial\varphi_I}{\partial x} = \frac{\partial\varphi_{II}}{\partial x}$$

$$\text{at } x=L, \varphi_{II} = \varphi_{III} \text{ and } \frac{\partial\varphi_I}{\partial x} = \frac{\partial\varphi_{II}}{\partial x}$$

We can do this:

$$1: A + B = C + D \qquad 2: ik_1A - ik_1B = ik_2C - ik_2D$$

$$3: Ce^{ik_2L} + De^{-ik_2L} = Fe^{ik_1L} \qquad 4: ik_2Ce^{ik_2L} - ik_2De^{-ik_2L} = ik_1Fe^{ik_1L}$$

And, of course we require:

$$\frac{BB^*}{AA^*} + \frac{FF^*}{AA^*} = 1$$

here are the results of applying the boundary conditions:

$$T = \frac{1}{\left[1 + \frac{v_0^2 \sin^2(k_{II}L)}{4E(E-V_0)}\right]}$$

$$R = 1 - T = T = \frac{\left[\frac{v_0^2 \sin^2(k_{II}L)}{4E(E-V_0)}\right]}{\left[1 + \frac{v_0^2 \sin^2(k_{II}L)}{4E(E-V_0)}\right]} = \frac{1}{\left[\frac{4E(E-V_0)}{V_0^2 \sin^2(k_{II}L)} + 1\right]}$$

Notice that the transmission probability is one when $k_{II}L = n\pi$. The meaning of this? When particles travel, they are reflected off of both faces of the barrier. At these special energies, the reflections from the two faces are exactly out of phase so that the reflected waves cancel.

Potential Barrier with E<V0

The ghost operator ...

Using very similar arguments shown previously

$$T = \frac{1}{1 + \frac{V_0^2 \sinh^2(\kappa L)}{4E(V_0 - E)}}; \kappa \equiv \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

The reflection coefficient is then 1-T.

I'm going to plot this ... but we need to do something to reduce the number of variables. Let $E=bV_0$. Then:

$$T = \frac{1}{1 + \frac{\sinh^2(\kappa L)}{4b(1-b)}}; \kappa \equiv \frac{\sqrt{2mV_0(1-b)}}{\hbar}$$

I've also represented v in terms of the log of the number of particle masses in the well. The plot shows all the important details.