Relativistic Acceleration Transformation

In answer to JR's question regarding accelerations

Consider Mary in a frame moving along the x axis with a velocity v relative to Frank. In the moving frame, a particle moving with a velocity u' has an acceleration a'. What does Frank observe?

The Lorentz transformations are:

$$x = \gamma(x' + vt') : y = y' : z = z' : t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\beta^2}}$$

$$x' = \gamma(x - vt) : y' = y : z' = z : t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-\beta^2}}$$

Let me get the velocity transforms first.

$$x = \gamma(x' + vt') : y = y' : z = z' : t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\beta^2}}$$

$$\Rightarrow \frac{dx}{dt} = u_x = \frac{\gamma(dx' + v\,dt')}{\frac{dt' + \frac{v}{c^2}dx'}{\sqrt{1-\beta^2}}} = \left[\frac{u_x' + v}{1 + \frac{u_x' v}{c^2}}\right]$$

$$\Rightarrow \frac{dy}{dt} = u_y = \frac{dy'}{dt'} = \frac{dy'}{\frac{dt' + \frac{v}{c^2}dx'}{\sqrt{1-\beta^2}}} = \sqrt{1-\beta^2}\,\frac{\frac{dy'}{dt'}}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \sqrt{1-\beta^2}\,\frac{u_y'}{1 + \frac{v}{c^2}u_x'} = \frac{\frac{u_y'}{\gamma}}{1 + \frac{v}{c^2}u_x'}$$

$$\Rightarrow \frac{dz}{dt} = u_z = \frac{\frac{u_z'}{\gamma}}{1 + \frac{v}{c^2}u_x'}$$

Now to obtain the acceleration transformations, we take the differential increments again. The resulting transformations (as will be shown below) are given by:

$$a_x = \frac{a_x'}{\gamma^3\left[1 + \frac{u_x' v}{c^2}\right]^3}$$

$$a_y = \frac{1}{\gamma^2\left[1 + \frac{v}{c^2}u_x'\right]^2}\left[a_y' - a_x' \frac{u_y'}{\left[1 + \frac{v}{c^2}u_x'\right]}\frac{v}{c^2}\right]$$

$$a_z = \frac{1}{\gamma^2\left[1 + \frac{v}{c^2}u_x'\right]^2}\left[a_z' - a_x' \frac{u_z'}{\left[1 + \frac{v}{c^2}u_x'\right]}\frac{v}{c^2}\right]$$

Derivation of acceleration transformations:

$$x = \gamma\left(x' + vt'\right) : y = y' : z = z' : t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\beta^2}}$$

$$\Rightarrow a_x = \frac{u_x}{dt} = \frac{\left[\frac{du_x'}{1+\frac{u_x' v}{c^2}}\right] - \left[\frac{u_x' + v}{\left(1+\frac{u_x' v}{c^2}\right)^2}\right]\frac{v}{c^2}du_x'}{\frac{dt' + \frac{v}{c^2}dx'}{\sqrt{1-\beta^2}}} = \sqrt{1-\beta^2}\,\frac{\left[\frac{du_x'}{1+\frac{u_x' v}{c^2}}\right] - \left[\frac{u_x' + v}{\left(1+\frac{u_x' v}{c^2}\right)^2}\right]\frac{v}{c^2}du_x'}{dt' + \frac{v}{c^2}dx'}$$

$$= \sqrt{1-\beta^2}\,\frac{\left[\frac{a_x'}{1+\frac{u_x' v}{c^2}}\right] - \left[\frac{u_x' + v}{\left(1+\frac{u_x' v}{c^2}\right)^2}\right]\frac{v}{c^2}a_x'}{1+\frac{v}{c^2}u_x'} = \frac{a_x'}{\left[1+\frac{u_x' v}{c^2}\right]^2}\sqrt{1-\beta^2}\left[1 - \frac{v}{c^2}\,\frac{u_x' + v}{\left(1+\frac{u_x' v}{c^2}\right)}\right]$$

$$= \frac{a_x'}{\left[1+\frac{u_x' v}{c^2}\right]^2}\sqrt{1-\beta^2}\left[\frac{\left(1+\frac{u_x' v}{c^2}\right)}{\left(1+\frac{u_x' v}{c^2}\right)} - \frac{v}{c^2}\,\frac{u_x' + v}{\left(1+\frac{u_x' v}{c^2}\right)}\right]$$

$$= \frac{a_x'}{\left[1+\frac{u_x' v}{c^2}\right]^2}\sqrt{1-\beta^2}\left[\frac{\left(1+\frac{u_x' v}{c^2} - \frac{vu_x'}{c^2} - \beta^2\right)}{\left(1+\frac{u_x' v}{c^2}\right)}\right] = \frac{a_x'}{\left[1+\frac{u_x' v}{c^2}\right]^3}\left[1-\beta^2\right]^{3/2}$$

$$\Rightarrow a_x = \frac{a_x'}{\gamma^3\left[1+\frac{u_x' v}{c^2}\right]^3}$$

$$a_y = \frac{du_y}{dt} = \frac{\frac{1}{\gamma}\frac{du_y'}{1+\frac{v}{c^2}u_x'} - \frac{1}{\gamma}\frac{u_y'}{\left[1+\frac{v}{c^2}u_x'\right]^2}\frac{v}{c^2}du_x'}{\frac{dt' + \frac{v}{c^2}dx'}{\sqrt{1-\beta^2}}} = \frac{1}{\gamma}\sqrt{1-\beta^2}\,\frac{\frac{a_y'}{1+\frac{v}{c^2}u_x'} - \frac{u_y'}{\left[1+\frac{v}{c^2}u_x'\right]^2}\frac{v}{c^2}a_x'}{1+\frac{v}{c^2}u_x'}$$

$$= \frac{1}{\gamma}\frac{\sqrt{1-\beta^2}}{\left[1+\frac{v}{c^2}u_x'\right]^2}\left[a_y' - a_x'\,\frac{u_y'}{\left[1+\frac{v}{c^2}u_x'\right]}\,\frac{v}{c^2}\right] = \frac{1}{\gamma^2\left[1+\frac{v}{c^2}u_x'\right]^2}\left[a_y' - a_x'\,\frac{u_y'}{\left[1+\frac{v}{c^2}u_x'\right]}\,\frac{v}{c^2}\right]$$

$$\Rightarrow a_z = \frac{1}{\gamma^2\left[1+\frac{v}{c^2}u_x'\right]^2}\left[a_z' - a_x'\,\frac{u_z'}{\left[1+\frac{v}{c^2}u_x'\right]}\,\frac{v}{c^2}\right]$$

The transformations for the accelerations are not nearly as clean-cut as for the velocities, but one important point is shown here: where-as in the classical mechanics, accelerations are orthogonal along the three Cartesian coordinates when transformed, under special relativity this is not the case.

If the particle is moving along the x-axis only or if the acceleration is zero in the x-direction, these transformations reduce to become:

$$a_x = \frac{a_x'}{\gamma^3\left[1+\frac{u_x' v}{c^2}\right]^3} : a_y = \frac{a_y'}{\gamma^2\left[1+\frac{v}{c^2}u_x'\right]^2} : a_z = \frac{a_z'}{\gamma^2\left[1+\frac{v}{c^2}u_x'\right]^2}$$