

Geometrical Optics

These notes show you how the various important equations that we have used are obtained. You will not be expected to be able to derive these results on a test.

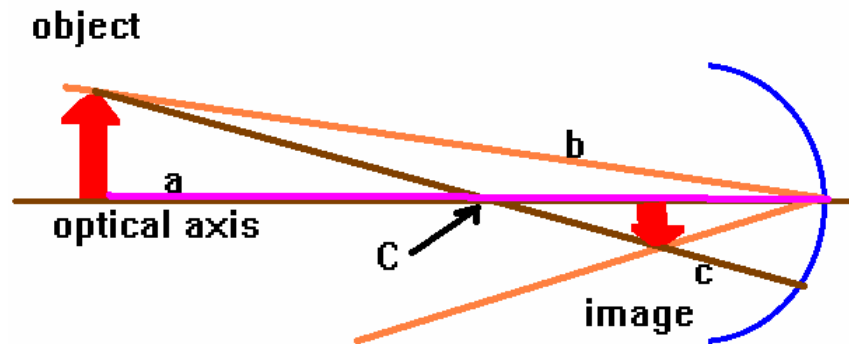
Derivation of

- (1) the definition for magnification
- (2) the mirror equation and the focal length of a mirror
- (3) the magnification of a refracting surface
- (4) the thin lens equation, and
- (5) the lens makers' equation for the focal length of a lens.

I: How to do optical ray tracing (for mirrors)

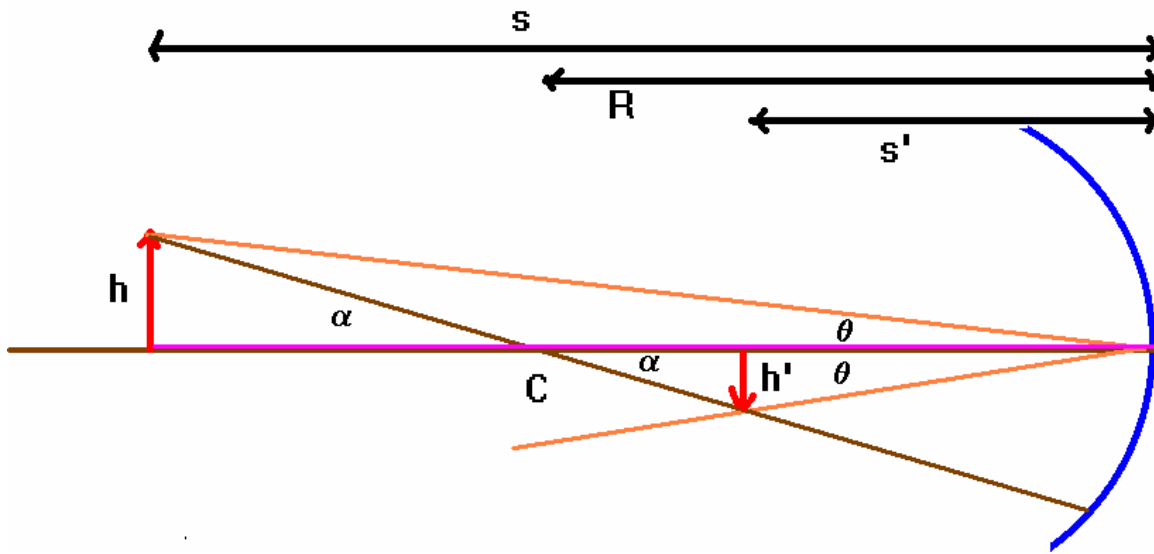
- (a) Let the first ray leave the bottom of the object and travel along the optical axis. {the optical axis is the symmetry axis of the mirror}
 - (b) Let the second ray leave the top of the object and strike the mirror where the optical axis touches the mirror.
 - (c) Let the third ray leave the top of the object and pass through the center of curvature.
- Of these three rays, only the second ray will not be reflected back upon itself.

The image will form at the intersection of rays b and c as shown.



[I have made an animated gif to show the parts involved in this raytracing.](#)

II: How does ray tracing lead to the mirror equation?



First let's show that

$$M \equiv \frac{h'}{h} = -\frac{s'}{s}$$

Look at the angles marked θ in the figure above.
The law of reflection says that the two angles are the same. Then:

$$\tan(\theta) = \frac{h}{s} \text{ and } \tan(\theta) = -\frac{h'}{s'}$$

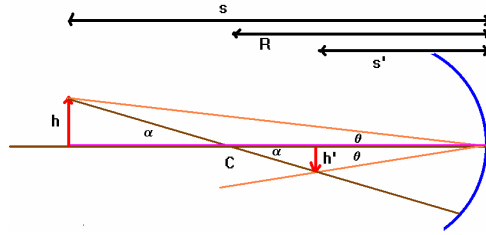
So we then have the result

$$\frac{h}{s} = -\frac{h'}{s'}$$

We see then that

$$\frac{h'}{h} = -\frac{s'}{s} \equiv M$$

Now let me show you where the mirror equation comes from.



$$\tan(\alpha) = \frac{h}{s-R} \quad \text{and} \quad \tan(\alpha) = -\frac{h'}{R-s'}$$

Divide these two equations:

$$\frac{\tan(\alpha)}{\tan(\alpha)} = 1 = \frac{\frac{h}{s-R}}{-\frac{h'}{R-s'}} = -\frac{h}{h'} \frac{R-s'}{s-R} = \frac{s(R-s')}{s'(s-R)}$$

We can show that this reduces to the mirror equation as follows:

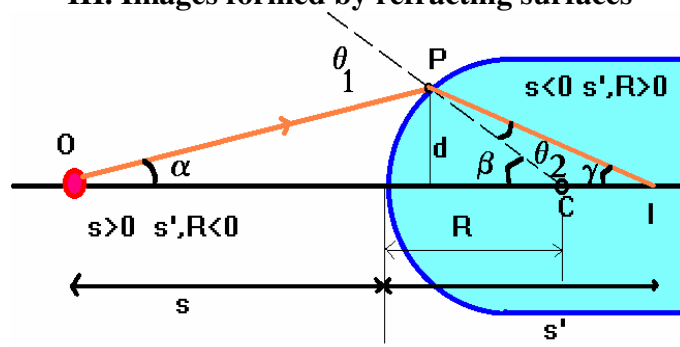
$$1 = \frac{s(R-s')}{s'(s-R)} \Rightarrow s's - s'R = sR - ss' \text{ so:}$$

$$2ss' = (s + s')R \Rightarrow \frac{ss'}{s+s'} = \frac{R}{2} \Rightarrow \frac{s+s'}{ss'} = \frac{2}{R} = \frac{1}{s} + \frac{1}{s'}$$

If we define the focal length as R/2 then the mirror equation results.

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \quad \text{where } f = \frac{R}{2}$$

III. Images formed by refracting surfaces



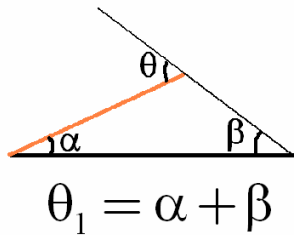
Snell's law tells us how θ_1 and θ_2 are related: $n_1 \sin \theta_1 = n_2 \sin \theta_2$

Paraxial approximation: the rays are parallel to the optical axis and not far from it.

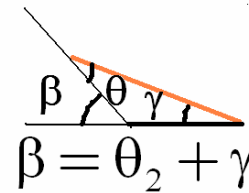
If you assume paraxial rays, then Snell's law is approximately given by

$$n_1 \theta_1 = n_2 \theta_2$$

Geometry \Rightarrow "exterior angle of any triangle = the sum of the two opposite interior angles"



$$\theta_1 = \alpha + \beta$$



$$\beta = \theta_2 + \gamma$$

$$\theta_1 = \alpha + \beta \text{ and } \beta = \theta_2 + \gamma$$

combine this result with the approximated form of Snell's law:

$$n_1(\alpha + \beta) = n_2(\beta - \gamma) \Rightarrow n_1\alpha + n_1\beta = n_2\beta - n_2\gamma$$

$$\text{This simplifies to: } n_1\alpha + n_2\gamma = (n_2 - n_1)\beta$$

Trigonometry (and small angles) \Rightarrow

$$\tan \alpha = \frac{d}{s} \approx \alpha, \quad \tan \beta = \frac{d}{R} \approx \beta, \quad \tan \gamma = \frac{d}{s'} \approx \gamma$$

Use these results in the *simplified Snell's law*:

$$\frac{n_1 d}{s} + \frac{n_2 d}{s'} = \frac{d(n_2 - n_1)}{R}$$

One more simplification: divide by d to get:

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{(n_2 - n_1)}{R}$$

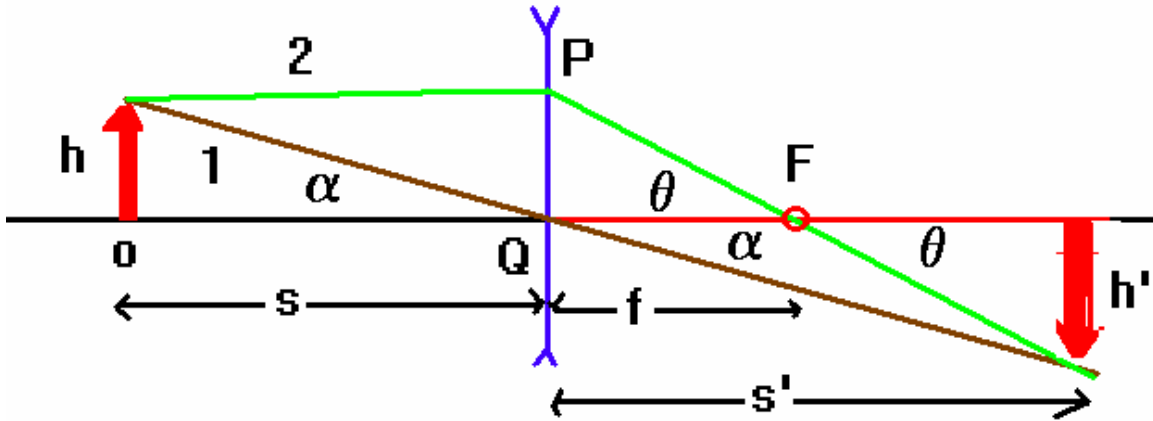
This equation above tells us where the image forms for a single refracting surface.

Here is an important note about the magnification:

The magnification for a refracting surface such as this *is different* from what we have had before. It is given by

$$M = -\frac{n_1 s'}{n_2 s}$$

IV. The thin lens equation



Note the rays marked 1 and 2 above.

This also shows you how to do raytracing with lenses.

ray 1 passes through the vertex of the lens and is not deviated.

ray 2 is parallel to the optical axis and thus will pass through the focal point of the lens.

Intersect these two rays (or their extensions) to get the image location.

[I have made an animation showing raytracing for lenses.](#)

$\tan \alpha = \frac{h}{s}$ and $\tan \alpha = -\frac{h'}{s'}$. Thus $M \equiv \frac{h'}{h} = -\frac{s'}{s}$ as with mirrors.

now, look at the angles marked θ (they are both the same because of intersecting lines)

$$\tan \theta = \frac{PQ}{f} = \frac{h}{f} \text{ and } \tan \theta = -\frac{h'}{s'-f}$$

thus

$$\frac{h}{f} = -\frac{h'}{s'-f} \Rightarrow \frac{h'}{h} = \frac{s'-f}{f} = \frac{s'}{s} \text{ (using the result for } M \text{ above)}$$

Let's cast this into the form of the thin lens equation by dividing by s' :

$$\frac{s'-f}{f} = \frac{s'}{s} \Rightarrow \frac{1-f}{f} = \frac{1}{s} \Rightarrow \frac{1}{f} - \frac{1}{s'} = \frac{1}{s}$$

Thus you see the thin lens equation results:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

V. Where does the lensmaker's equation come from?

Recall how images behave at refracting surfaces:

$$\frac{n_1}{s_1} + \frac{n_2}{s'_1} = \frac{(n_2 - n_1)}{R_1} \Rightarrow \frac{n_2}{s'_1} = \frac{(n_2 - n_1)}{R_1} - \frac{n_1}{s_1}$$

The original object is at s_1 and the first image is at s'_1 .

Imagine a thin lens surrounded by a medium of index of refraction n_1 .

Now let's form a second surface and look for the image:

$$\frac{n_2}{s_2} + \frac{n_1}{s'_2} = \frac{(n_1 - n_2)}{R_2} = -\frac{(n_2 - n_1)}{R_2}$$

The (second) object is at s_2 and the (second) image is at s'_2 .

Since the lens is thin, the image from the first side forms the object for the second side but it is virtual. Thus $s_2 = -s'_1$ **if the lens is not thin, this needs to be modified.**

I use this in the second equation to give:

$$\frac{n_2}{s_2} + \frac{n_1}{s'_2} = -\frac{(n_2 - n_1)}{R_2} \Rightarrow -\frac{n_2}{s'_1} + \frac{n_1}{s'_2} = -\frac{(n_2 - n_1)}{R_2}$$

Now I use the results from the first equation and obtain:

$$\begin{aligned} \frac{n_1}{s_1} + \frac{n_2}{s'_1} &= \frac{(n_2 - n_1)}{R_1} \Rightarrow \frac{n_2}{s'_1} = \frac{(n_2 - n_1)}{R_1} - \frac{n_1}{s_1} \\ -\frac{n_2}{s'_1} + \frac{n_1}{s'_2} &= -\frac{(n_2 - n_1)}{R_2} \Rightarrow -\left[\frac{(n_2 - n_1)}{R_1} - \frac{n_1}{s_1}\right] + \frac{n_1}{s'_2} = -\frac{(n_2 - n_1)}{R_2} \end{aligned}$$

A little bit of algebra:

$$\frac{n_1}{s_1} + \frac{n_1}{s'_2} = -\frac{(n_2 - n_1)}{R_2} + \left[\frac{(n_2 - n_1)}{R_1}\right] = (n_2 - n_1) \left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

Divide by n_1 to obtain:

$$\frac{n_1}{s_1} + \frac{n_1}{s'_2} = (n_2 - n_1) \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \Rightarrow \frac{1}{s} + \frac{1}{s'} = \left(\frac{n_2}{n_1} - 1\right) \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Then recognize that $s_1 = s$ and $s'_2 = s'$ that we used in the thin lens equation. Then we have:

$$\frac{1}{s} + \frac{1}{s'} = \left(\frac{n_2}{n_1} - 1\right) \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

If the lens is in air, then $n_1 = 1$ so the lensmaker's equation becomes

$$\frac{1}{s} + \frac{1}{s'} = (n - 1) \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

You can't help but notice that this is the focal length in the thin lens equation. Thus

$$\frac{1}{f} = (n - 1) \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

and the thin lens equation is then:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$