Magnetic Field from a wire solved 3 ways.

$$\oint \vec{B} \bullet d\vec{l} = \mu_0 I_c$$

Choose a wire running a current along the +z direction. Then according to the right hand rule:

$$\vec{B} = B\hat{\theta}$$

In terms of Cartesian coordinates, we write the cylindrical angle unit vector as:

$$\hat{\theta} = -\sin(\theta)\hat{x} + \cos(\theta)\hat{y}$$

Proof:

We want a right handed coordinate system in cylindrical coordinates so that:

$$\hat{r} \times \hat{\theta} = \hat{z}$$

The radial unit vector is given by:

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y}$$

And so:

$$\hat{r} \times \hat{\theta} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos\theta & \sin\theta & 0 \\ \theta_x & \theta_y & 0 \end{vmatrix} = \hat{z}\left[\theta_y \cos\theta - \theta_x \sin\theta\right] \Rightarrow \left[\theta_y \cos\theta - \theta_x \sin\theta\right] = 1$$

An obvious solution would be:

$$\theta_x = -\sin\theta : \theta_y = \cos\theta \Rightarrow \hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y}$$

It is clear that this gives zero projection along the z axis and also it has a magnitude of 1.

Thus this is the desired unit vector.

Now we then have:

$$\vec{B} \bullet d\vec{l} = \left[B_x\hat{x} + B_y\hat{y}\right] \bullet \left[dx\hat{x} + dy\hat{y}\right]$$

to find the x and y components of B, you do the usual operation:

$$\vec{B} \bullet \hat{x} = B_x = B\hat{\theta} \bullet \hat{x} = -B\sin\theta : \vec{B} \bullet \hat{y} = B_y = B\hat{\theta} \bullet \hat{y} = B\cos\theta$$

Thus:

$$\vec{B} \bullet d\vec{l} = -B\sin\theta dx + B\cos\theta dy$$

Now, on a circular path of radius R, we have:

$$d\vec{l} = dx\hat{x} + dy\hat{y} = \left[-R\sin\theta\hat{x} + R\cos\theta\hat{y}\right]d\theta$$

This then gives:

$$\vec{B} \bullet d\vec{l} = BRd\theta \Rightarrow \oint \vec{B} \bullet d\vec{l} = BR\int_{\theta=0}^{\theta=2\pi} d\theta = 2\pi RB$$

Now apply Ampere's law:

$$\oint \vec{B} \bullet d\vec{l} = \mu_0 I_c \Rightarrow B(2\pi R) = \mu_0 I \Rightarrow \vec{B} = \tfrac{\mu_0 I}{2\pi R}\hat{\theta}$$

This is for an infinitely long wire carrying a current I in the +z direction.

Now let's calculate the same via Biot-Savert:

$$d\vec{B} = \frac{\mu_0 I d\vec{l}_i \times \hat{r}_{ip}}{4\pi \left|\vec{r}_{ip}\right|^2}$$

We calculate this above the symmetric center of the wire. Again let the wire run along the z axis with current going in the +z direction. Let's calculate it say at a point $\vec{r}_p$ in the xy plane. Then:

$$d\vec{l}_i = dz_i \hat{z} : \vec{r}_p = x_p \hat{x} + y_p \hat{y} : \vec{r}_i = z_i \hat{z} \Rightarrow \vec{r}_{ip} = x_p \hat{x} + y_p \hat{y} - z_p \hat{z}$$

Then:

$$d\vec{l}_i \times \vec{r}_{ip} = dz_i \left[\hat{z}_i \times \left(x_p \hat{x} + y_p \hat{y}\right)\right] = dz_i \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ x_p & y_p & 0 \end{vmatrix} = dz_i \left[-y_p \hat{x} + x_p \hat{y}\right]$$

so the differential magnetic field is:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{dz_i \left[\frac{-y_p \hat{x} + x_p \hat{y}}{\sqrt{x_p^2 + y_p^2 + z_i^2}}\right]}{\left[x_p^2 + y_p^2 + z_i^2\right]} = \frac{\mu_0 I}{4\pi} \frac{\left[-y_p \hat{x} + x_p \hat{y}\right]}{\left[x_p^2 + y_p^2 + z_i^2\right]^{3/2}} dz_i$$

For a finite wire stretching from -a to +a we have:

$$\int_{z_i=-a}^{z_i=+a} \frac{dz_i}{\left[x_p^2 + y_p^2 + z_i^2\right]^{3/2}} = \left.\frac{z_i}{\left(x_p^2 + y_p^2\right)\sqrt{x_p^2 + y_p^2 + z_i^2}}\right|_{-a}^{a} = \frac{2a}{\left(x_p^2 + y_p^2\right)\sqrt{x_p^2 + y_p^2 + a^2}}$$

Now let's recognize that the polar radial vector is given by:

$$\vec{r}_p = x_p \hat{x} + y_p \hat{y} \Rightarrow \left|\vec{r}_p\right| = \sqrt{x_p^2 + y_p^2}$$

Then the magnetic field is given by:

$$\vec{B} = \frac{\mu_0 I}{4\pi}\left[-y_p \hat{x} + x_p \hat{y}\right]\left\{\frac{2a}{r^2 \sqrt{r^2 + a^2}}\right\}$$

We can simplify further:

$$\vec{B} = \frac{\mu_0 I}{4\pi}\left[-y_p \hat{x} + x_p \hat{y}\right]\left\{\frac{2a}{r^2 \sqrt{r^2 + a^2}}\right\} = \frac{\mu_0 I}{4\pi r} 2a \frac{-\sin\theta \hat{x} + \cos\theta \hat{y}}{\sqrt{r^2 + a^2}} = \frac{\mu_0 I}{2\pi r} \frac{a}{\sqrt{r^2 + a^2}} \hat{\theta}$$

Now in the limit as the wire becomes infinite, we have:

$$\lim_{a \to \infty} \frac{a}{\sqrt{r^2 + a^2}} = 1$$

So the magnetic field from an infinitely long wire by direct integration is:

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\theta}$$

Finally done using Ampere's law but in a simpler viewpoint:
On the circular path, B is constant in direction and magnitude and is parallel to the differential path:

$$\oint \vec{B} \bullet d\vec{l} = B \oint dl = B \int \sqrt{dx^2 + dy^2} = B \int \sqrt{r^2 \left(-\sin\theta\right)^2 \left[d\theta\right]^2 + r^2 \left(\cos\theta\right)^2 \left[d\theta\right]^2}$$

So:

$$\oint \vec{B} \bullet d\vec{l} = Br \int_{\theta=0}^{\theta=2\pi} d\theta = 2\pi r B$$

Then:

$$2\pi r B = \mu_0 I_c \Rightarrow \vec{B} = \frac{\mu_o I}{2\pi r}\hat{\theta}$$