Problem 1.5

To show: $\vec{A}x\left(\vec{B}x\vec{C}\right)=\vec{B}\left(\vec{A}\bullet\vec{C}\right)-\vec{C}\left(\vec{A}\bullet\vec{B}\right)$

$$\vec{B}x\vec{C}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}=\left(B_yC_z-B_zC_y\right)\hat{i}-\left(B_xC_z-B_zC_x\right)\hat{j}+\left(B_xC_y-B_yC_x\right)\hat{k}$$

$$\vec{A}x\left(\vec{B}x\vec{C}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ \left(\vec{B}\times\vec{C}\right)_x & \left(\vec{B}\times\vec{C}\right)_y & \left(\vec{B}\times\vec{C}\right)_z\end{vmatrix}$$

$$=\left[A_y\left(\vec{B}\times\vec{C}\right)_z-A_z\left(\vec{B}\times\vec{C}\right)_y\right]\hat{i}-\left[A_x\left(\vec{B}\times\vec{C}\right)_z-A_z\left(\vec{B}\times\vec{C}\right)_x\right]\hat{j}+\left[A_x\left(\vec{B}\times\vec{C}\right)_y-A_y\left(\vec{B}\times\vec{C}\right)_x\right]\hat{k}$$

$$=\left[A_y\left(B_xC_y-B_yC_x\right)+A_z\left(B_xC_z-B_zC_x\right)\right]\hat{i}$$
$$-\left[A_x\left(B_xC_y-B_yC_x\right)-A_z\left(B_yC_z-B_zC_y\right)\right]\hat{j}$$
$$+\left[-A_x\left(B_xC_z-B_zC_x\right)-A_y\left(B_yC_z-B_zC_y\right)\right]\hat{k}$$
$$=\left[A_y\left(B_xC_y-B_yC_x\right)+A_z\left(B_xC_z-B_zC_x\right)\right]\hat{i}$$
$$-\left[A_x\left(B_xC_y-B_yC_x\right)-A_z\left(B_yC_z-B_zC_y\right)\right]\hat{j}$$
$$+\left[-A_x\left(B_xC_z-B_zC_x\right)-A_y\left(B_yC_z-B_zC_y\right)\right]\hat{k}$$
$$=$$
$$+B_x\left(A_yC_y+A_zC_z+A_xC_x\right)\hat{i}-A_xB_xC_x\hat{i}-C_x\left(A_yB_y+A_zB_z+A_xB_x\right)\hat{i}+A_xB_xC_x\hat{i}$$
$$+B_y\left(A_xC_x+A_zC_z+A_yC_y\right)\hat{j}-A_yB_yC_y\hat{j}-C_y\left(A_xB_x+A_zC_z+A_yB_y\right)\hat{j}+A_yB_yC_y\hat{j}$$
$$+B_z\left(A_xC_x+A_yC_y+A_zC_z\right)\hat{k}-A_zB_zC_z\hat{k}-C_z\left(A_xB_x+A_yB_y+A_zB_z\right)\hat{k}+A_zB_zC_z\hat{k}$$
$$=\vec{B}\left(\vec{A}\bullet\vec{C}\right)-\vec{C}\left(\vec{A}\bullet\vec{B}\right)$$

Thus

$$\vec{A}x\left(\vec{B}x\vec{C}\right)=\vec{B}\left(\vec{A}\bullet\vec{C}\right)-\vec{C}\left(\vec{A}\bullet\vec{B}\right)$$

All higher vector products can be reduced by repeated application of the bac-cab rule.

Position, Displacement, and separation vectors
Important notations and distinctions

Position Vector: points from the origin to a position in space

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

Magnitude of the position vector

$$|\vec{r}| = \sqrt{\vec{r} \bullet \vec{r}} = \sqrt{x^2 + y^2 + z^2}$$

Unit position vector

$$\hat{r} \equiv \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}}$$

field point: $\vec{r}$ source point: $\vec{r}'$

The **separation vector** between a source point and a field point is given by:

$$\vec{\mathfrak{R}} = \vec{r} - \vec{r}\,'$$

You can remember this by remembering one is a field point and one is a source point. f comes before s in the alphabet.

This is the vector pointing from the source point towards the field point.

The **unit separation vector** is

$$\hat{\mathfrak{R}} \equiv \frac{\vec{\mathfrak{R}}}{|\vec{\mathfrak{R}}|} = \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}$$

and points in the direction from the source point towards the field point.

In Cartesian Coordinates:

$$\vec{\mathfrak{R}} = (x - x')\,\hat{x} + (y - y')\,\hat{y} + (z - z')\,\hat{k}$$

The magnitude of the separation vector is:

$$\mathfrak{R} \equiv |\vec{\mathfrak{R}}| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

The unit separation vector is given by:

$$\hat{\mathfrak{R}} = \frac{(x-x')\hat{x}+(y-y')\hat{y}+(z-z')\hat{k}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

Problem 1.7: Find the separation vector $\vec{\mathfrak{R}}$ from the source point (2,8,7) to the field point (4,6,8). Determine the magnitude and the unit vector.

$$\vec{\mathfrak{R}} = (4-2)\,\hat{x} + (6-8)\,\hat{y} + (8-7)\,\hat{z} = +2\hat{x} - 2\hat{y} + 1\hat{z}$$

$$\left|\vec{\mathfrak{R}}\right| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$$

$$\hat{\mathfrak{R}} = +\tfrac{2}{3}\,\hat{x} - \tfrac{2}{3}\,\hat{y} + \tfrac{1}{3}\,\hat{z}$$

Although your author defines this type of vector, I find this extremely difficult for students to constantly use and keep straight from other vectors. Therefore, I choose to use this notation:

$$\vec{r}_{ip} \equiv \vec{r}_p - \vec{r}_i$$

is the vector pointing from point i towards point p. I choose point i since this is typically a charge (the ith. charge) while p is a point in space. The unit vector is defined by:

$$\hat{r}_{ip} \equiv \frac{\vec{r}_p - \vec{r}_i}{\left|\vec{r}_p - \vec{r}_i\right|} = \frac{\vec{r}_{ip}}{\left|\vec{r}_{ip}\right|}$$

Differential Calculus

Infinitesimal Displacement Vector

$$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

This is pretty much the same as $d\vec{r}$.

Ordinary derivative of a single variable:

Consider f=f(x) only. Then $df = \left(\frac{df}{dx}\right)dx$

Interpretation: slope of the curve.

Gradient:

Suppose T=T(x,y,z)

then,

$$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz$$

This can be written in vector notation:

$$dT = \left(\frac{\partial T}{\partial x}\,\hat{x} + \frac{\partial T}{\partial y}\,\hat{y} + \frac{\partial T}{\partial z}\,\hat{z}\right) \bullet \left(dx\hat{x} + dy\hat{y} + dz\hat{z}\right)$$

or

$$dT = \left(\vec{\nabla} T\right) \bullet d\vec{l}$$

where

$$\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

the application of the gradient to a function produces a vector and its application to a function points in the direction of the maximum increase of the function.

The magnitude of the application of the gradient gives the slope along this direction.

Let's now do some in-class problems.
Problem 1.11: find the gradients of the following functions:

$$(a) f(x, y, z) = x^2 + y^3 + z^4$$

$$(b) f(x, y, z) = x^2 y^3 z^4$$

$$(c) f(x, y, z) = e^x \sin(y) \ln(z)$$

Important: make sure you can do partial derivatives!

Problem 1.13: Let $\vec{\Re}$ be the separation vector from a fixed point (x',y',z') (source point) to the point (x,y,z) (field point), and let $\Re$ be its length. Show that:

$$(a) \vec{\nabla}\left(\Re^2\right) = 2\vec{\Re}$$

$$(b) \vec{\nabla}\left(\tfrac{1}{\Re}\right) = -\tfrac{\vec{\Re}}{\Re^2}$$

(c) what is the general formula for $\vec{\nabla}\left(\Re^n\right)$?

There are at least 3 ways (actually more if you consider multiple applications) that the del operator can act on functions. Here they are:

(1) Gradient: $\vec{\nabla}T$ (produces a vector function)

(2) divergence: $\vec{\nabla} \bullet \vec{T}$ (produces a scalar function)

(3) curl: $\vec{\nabla} \times \vec{T}$ (produces something that looks a lot like a vector).

Divergence operations

Suppose you have a vector function:

$$\vec{T} = T_x\hat{x} + T_y\hat{y} + T_z\hat{z}$$

The divergence of T is given by:

$$\vec{\nabla} \bullet \vec{T} = \frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + \frac{\partial T_z}{\partial z}$$

The **divergence** of a vector is a scalar and it is a measure of how much the vector spreads out (diverges) from the point in question.

Problem 1.15: Calculate the divergence of the following vector functions:

$$(1)\,\vec{V}_a = x^2\hat{x} + 2xz^2\hat{y} - 2xz\hat{z}$$

$$(2)\vec{V}_b = xy\hat{x} + 2yz\hat{z} + 3zx\hat{z}$$

$$(3)\vec{V}_c = y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}$$

The Curl

The definition of the curl of a vector function:

$$\vec{T} = T_x\hat{x} + T_y\hat{y} + T_z\hat{z}$$

is given by:

$$\vec{\nabla}\times\vec{T} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ T_x & T_y & T_z \end{vmatrix} = \left(\frac{\partial T_z}{\partial y} - \frac{\partial T_y}{\partial z}\right)\hat{x} - \left(\frac{\partial T_z}{\partial x} - \frac{\partial T_x}{\partial z}\right)\hat{y} + \left(\frac{\partial T_y}{\partial x} - \frac{\partial T_x}{\partial y}\right)\hat{z}$$

The curl of a vector is a vector function (I am including the class of pseudovectors here) and the geometrical interpretation of the curl is to tell how much the vector function "curls around" the point in question.

Problem 1.18: Calculate the curl of the following vector functions:

$$(1)\,\vec{V}_a = x^2\hat{x} + 2xz^2\hat{y} - 2xz\hat{z}$$

$$(2)\vec{V}_b = xy\hat{x} + 2yz\hat{z} + 3zx\hat{z}$$

$$(3)\vec{V}_c = y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}$$

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sum rules

$$\vec{\nabla}(f + g) = \vec{\nabla}f + \vec{\nabla}g$$

$$\vec{\nabla}\bullet\left(\vec{A}+\vec{B}\right) = \left(\vec{\nabla}\bullet\vec{A}\right) + \left(\vec{\nabla}\bullet\vec{B}\right)$$

$$\vec{\nabla}\times\left(\vec{A}+\vec{B}\right) = \left(\vec{\nabla}\times\vec{A}\right) + \left(\vec{\nabla}\times\vec{B}\right)$$

product rules

for k a constant:

$$\vec{\nabla}(kf) = k\vec{\nabla}f$$

$$\vec{\nabla}\bullet\left(k\vec{A}\right) = k\vec{\nabla}\bullet\vec{A}$$

$$\vec{\nabla}\times\left(k\vec{A}\right) = k\vec{\nabla}\times\vec{A}$$

vector product rules

(1) Gradient:

$$\vec{\nabla}(fg) = f\vec{\nabla}g + g\vec{\nabla}f$$

$$\vec{\nabla}\left(\vec{A} \bullet \vec{B}\right) = \vec{A}\times\left(\vec{\nabla}\times\vec{B}\right) + \vec{B}\times\left(\vec{\nabla}\times\vec{A}\right) + \left(\vec{A}\bullet\vec{\nabla}\right)\vec{B} + \left(\vec{B}\bullet\vec{\nabla}\right)\vec{A}$$

(2) Divergence

$$\vec{\nabla}\bullet\left(f\vec{A}\right) = f\left(\vec{\nabla}\bullet\vec{A}\right) + \vec{A}\bullet\left(\vec{\nabla}f\right)$$

$$\vec{\nabla}\bullet\left(\vec{A}\times\vec{B}\right) = \vec{B}\bullet\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}\bullet\left(\vec{\nabla}\times\vec{B}\right)$$

(3) Curls

$$\vec{\nabla}\times\left(f\vec{A}\right) = f\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}\times\left(\vec{\nabla}f\right)$$

$$\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right) = \left(\vec{B}\bullet\vec{\nabla}\right)\vec{A} - \left(\vec{A}\bullet\vec{\nabla}\right)\vec{B} + \vec{A}\left(\vec{\nabla}\bullet\vec{B}\right) - \vec{B}\left(\vec{\nabla}\bullet\vec{A}\right) \text{ LTO}$$

Your author suggests you paste these somewhere in your text.
Make sure you check to see that I've typed everything correctly!

Notice problem 1.22. We're not so we'll travel on.

We will need to worry about second derivatives. There are about 5 possibilities, here they are:

$$\vec{\nabla}\bullet\left(\vec{\nabla}T\right) = \tfrac{\partial^2 T}{\partial x^2} + \tfrac{\partial^2 T}{\partial y^2} + \tfrac{\partial^2 T}{\partial z^2} \text{ (the Laplacian of T) \& } \vec{\nabla}^2\vec{v} \equiv \left(\vec{\nabla}^2 v_x\right)\hat{x} + \left(\vec{\nabla}^2 v_y\right)\hat{y} + \left(\vec{\nabla}^2 v_z\right)\hat{z}$$

$$\vec{\nabla}\times\left(\vec{\nabla}T\right) = \vec{0} \text{ Please note text discussion below eq. 1.44}$$

$$\vec{\nabla}\left(\vec{\nabla}\bullet\vec{V}\right) \text{ does not, for some reason, regularly appear in physics}$$

$$\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{V}\right) = 0 \text{ Please note text discussion below eq. 1.46}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{V}\right) = \vec{\nabla}\left(\vec{\nabla}\bullet\vec{V}\right) - \nabla^2\vec{V} \text{ (no new operation overall)}$$

Interestingly enough, it thus turns out that the Laplacian is the most normally 2nd vector derivative which is encountered.

Integral Calculus

(1) Line integrals

A line integral is a vector function which is integrated over a specified path.
Here is one form:

$$\int_{\text{Path}} \vec{V} \bullet d\vec{l}$$

For a closed path:

$$\oint \vec{V} \bullet d\vec{l}$$

Normally, the value of the vector depends upon path but for CONSERVATIVE vectors, we will have path independence.

It is, however, worthwhile to work one example.
In the Cartesian coordinate system,

$$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

Your author in example 1.6 says that is what this is always. The path may be converted from Cartesian to other systems in order to make things easier.
I recommend that you follow your author's suggestion always.

Example 1.6:

$$\vec{V} = y^2\hat{x} + 2x(y+1)\hat{y}$$

Paths:

(1) a=(1,1,0)->b=(2,2,0) along hypotenuse

(2) a=(1,1,0)->b=(2,1,0)->c=(2,2,0)

(1):

$$\text{equation of path: } y=x \Rightarrow dy=dx \Rightarrow d\vec{l}=dx\hat{x}+dx\hat{y}+0\hat{z}$$

$$\vec{V} \bullet d\vec{l} = x^2dx + 2x(x+1)dx = (3x^2 + 2x)dx$$

$$\int_{\text{path}} \vec{V} \bullet d\vec{l} = \int_{x=1}^{x=2} (3x^2 + 2x)dx$$

(2) other paths:

y=1;x not constant; z=0:

$$d\vec{l} = dx\hat{x}; \int_{\text{path}} \vec{V} \bullet d\vec{l} = \int_{x=1}^{x=2} 1^2 dx$$

+

x=1; y not constant; z=0:

$$d\vec{l} = dy\hat{y}; \int_{\text{path}} \vec{V} \bullet d\vec{l} = \int_{y=1}^{y=2} 2^2 dy$$

It is extremely important that you do this correctly:
When you encounter a path integral such as:

$$\oint d\vec{l}$$

you must write the path displacement in Cartesian coordinates as:

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

Here is an example of what can go wrong:

and let the path be on a circle of radius a. It would seem to be the most natural thing to do this:

$$d\vec{l} = a d\theta\,\hat{\theta}$$

(how?)

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} : \hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} :$$

$$\hat{r}\times\hat{\theta} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & +\cos\theta & 0 \end{vmatrix} = \hat{x}[0] - \hat{y}[0] + \hat{z}\left[\cos^2\theta + \sin^2\theta\right] = \hat{z}$$

so you might consider doing this:

$$\oint d\vec{l} = \int_{\theta=0}^{\theta=2\pi} a\,d\theta\,\hat{\theta} = a[2\pi]\hat{\theta}$$

But this is wrong since the net displacement on a circle is zero. The correct way to do this is:

$$\oint d\vec{l} = a\oint[-\sin\theta\hat{x} + \cos\theta\hat{y}] = -\hat{x}a\int_{\theta=0}^{\theta=2\pi}\sin\theta d\theta + \hat{y}a\int_{\theta=0}^{\theta=2\pi}\cos\theta d\theta = -\hat{x}a[-\cos\theta]\Big|_0^{2\pi} + \hat{y}a[\sin\theta]\Big|_0^{2\pi} = \vec{0}$$

By the way: if you want to find the unit vector associated with the zero vector, you're going to have to do it in a different way that the way I've shown before.