How to generate curvilinear coordinates

Start with Cylindrical coordinates:

$$\vec{s} = x\hat{x} + y\hat{y} = s\hat{s}$$

We can write x and y in polar coordinates as:

$$x = s\cos\varphi : y = s\sin\varphi \Rightarrow \vec{s} = s\cos\varphi\,\hat{x} + s\sin\varphi\,\hat{y} = s\left[\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}\right]$$

By inspection, it is clear that the unit s vector is then:

$$\hat{s} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}$$

Now to calculate the φ vector, we require:

$$\hat{s} \bullet \hat{\varphi} = 0 \Rightarrow \varphi_x \cos\varphi + \varphi_y \sin\varphi = 0$$

You may observe from this that one solution is given by:

$$\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$$

There are other solutions but this is the one that gives a right handed coordinate system so that:

$$\hat{s} \times \hat{\varphi} = \hat{z}$$

This then gives us the unit cylindrical vectors:

$$\hat{s} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}$$

$$\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$$

$$\hat{z} = \hat{z}$$

Now to obtain the spherical unit vectors, recognize that s is a projection of r and φ has the same meaning so:

$$\hat{r} = \sin\theta\,\hat{s} + \cos\theta\,\hat{z} : \hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$$

Now in order to find the θ unit vector, you take:

$$\hat{r} \times \hat{\varphi} = \hat{\theta}$$

so that

$$\hat{\theta} = \cos(\theta)\cos(\varphi)\,\hat{x} + \cos(\theta)\sin(\varphi)\,\hat{y} - \sin(\theta)\,\hat{z}$$

Now it is straight-forward but somewhat tedious to write the Cartesian coordinates in terms of cylindrical or spherical unit vectors.