Physics Version

Spherical Coordinates: Gravitational Attraction

The transformation from Cartesian to Spherical Coordinates is:

$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

We form the Jacobian:

$$\left|\vec{J}\right| = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$\left|\vec{J}\right| = \begin{bmatrix} \sin\theta\cos\varphi\left[r^2\sin^2\theta\cos\varphi\right] \\ -r\cos\theta\cos\varphi\left[-r\sin\theta\cos\theta\cos\varphi\right] = r^2\left[\cos^2\varphi\sin\theta\right] + r^2\sin\theta\sin^2\varphi \\ r^2\sin\theta\sin^2\varphi \end{bmatrix} = r^2\sin\theta$$

This gives us the linear transformation :

$$dx\,dy\,dz \to r\sin\theta\,dr\,d\varphi\,d\theta$$

Example 1: Find the volume of the sphere:

$$x^2 + y^2 + z^2 = a^2$$

The integral is given by:

$$V = \int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi}\int_{r=0}^{r=a} r^2\sin\theta\,dr\,d\varphi\,d\theta = 2\pi\left[\tfrac{a^3}{3}\right]\left[-\cos\theta\right]\Big|_0^\pi = -2\pi\left[\tfrac{a^3}{3}\right]\left[-1-1\right] = \tfrac{4}{3}\pi a^3$$

be careful about the azimuthal limits so that you don't add up your sphere twice too many times.

Example 2

Find the centroid of the region bounded by the sphere $r = a$ and the cone $\theta = \alpha$

The volume of the solid is:

$$V = \int_{\theta=0}^{\theta=\alpha}\int_{r=0}^{r=a}\int_{\varphi=0}^{\varphi=2\pi} r^2\sin\theta\,d\theta\,dr\,d\varphi = 2\pi\tfrac{a^3}{3}\left[-\cos\theta\right]\Big|_0^\alpha = \tfrac{2\pi a^3}{3}\left[1-\cos\alpha\right]$$

The z coordinate of the centroid is given by:

$$\overline{z} = \tfrac{1}{V}\int_{\theta=0}^{\theta=\alpha}\int_{r=0}^{r=a}\int_{\varphi=0}^{\varphi=2\pi} zr^2\sin\theta\,d\theta\,dr\,d\varphi = \tfrac{3}{a^3[1-\cos\alpha]}\int_{\theta=0}^{\theta=\alpha}\tfrac{a^4}{4}\cos\theta\sin\theta\,d\theta = \tfrac{3a}{4[1-\cos\alpha]}\int_{\theta=0}^{\theta=\alpha}\sin\theta\left[d\left(\sin\theta\right)\right]$$

$$= \tfrac{3a}{4[1-\cos\alpha]}\tfrac{\sin^2\theta}{2}\Big|_0^\alpha = \tfrac{3\sin^2\alpha}{8[1-\cos\alpha]}a = \tfrac{3}{8}a\tfrac{[1-\cos\alpha][1+\cos\alpha]}{[1-\cos\alpha]} = \tfrac{3}{8}a\left[1+\cos\alpha\right]$$

Example 3: Done the right way.

The vector pointing to a differential mass element

$dm = \sigma r^2\sin\theta\,dr\,d\theta\,d\varphi$ is given by:

$$\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$$

where the Cartesian coordinates are the points of a mass point relative to the origin.

The vector pointing to a point located along the z-axis is given by:

$$\vec{r}_p = 0\hat{i} + 0\hat{j} + z_p\hat{k}$$

The gravitational attraction for a mass element and a mass m located at the point is given by:

$$d\vec{F} = G\frac{m(dm_i)}{\left[\sqrt{x_i^2+y_i^2+(z_p-z_i)^2}\right]^2}\left[\hat{r}_{ip}\right] = -G\frac{m(dm_i)}{\left[\sqrt{x_i^2+y_i^2+(z_p-z_i)^2}\right]^2}\frac{x_i\hat{i}+y_i\hat{j}+(z_i-z_p)\hat{k}}{\sqrt{x_i^2+y_i^2+(z_p-z_i)^2}} = -mG\frac{x_i\hat{i}+y_i\hat{j}+(z_i-z_p)\hat{k}}{\left[x_i^2+y_i^2+(z_p-z_i)^2\right]^{\frac{3}{2}}}(dm_i)$$

We are going to integrate this over a sphere of radius a. I'll worry about scaling the mass later to fit this.

$$\vec{F} = \int d\vec{F} = -m\sigma G\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=a}\int_{\varphi=0}^{\varphi=2\pi}\frac{x_i\hat{i}+y_i\hat{j}+(z_i-z_p)\hat{k}}{\left[x_i^2+y_i^2+(z_p-z_i)^2\right]^{\frac{3}{2}}}r^2\sin\theta\,dr\,d\theta d\varphi$$

For the rest of the solution to this problem, I'll refer you to a similar problem which appears on my pages. You may investigate the details there if you wish. Soon there will be a much easier way to work this type of problem!

http://www.lyon.edu/webdata/users/shutton/phys350-spring2004/sphereofcharge.doc

In that problem, the analysis is quite similar, only the constants are different.