**Maxwell's equations and TEM waves**

| | **non-calculus** | **calculus** |
|---|---|---|
| Gauss's Law for Electrostatics | $\sum_{surface} \vec{E} \bullet \vec{A} = \frac{Q_{enc}}{\varepsilon_0}$ | $\oiint \vec{E} \bullet \vec{n} dA = \frac{Q_{enc}}{\varepsilon_0}$ |
| Gauss's Law for Magentostatics | $\sum_{surface} \vec{B} \bullet \vec{A} = 0$ | $\oiint \vec{B} \bullet \vec{n} dA = 0$ |
| Ampere's Law | $\sum_{curve} \vec{B} \bullet \vec{S} = \mu_0 I_c + \varepsilon_0 \frac{\Delta \Phi_E}{\Delta t}$ | $\oint_c \vec{B} \bullet d\vec{s} = \mu_0 I_c + \varepsilon_0 \frac{d\Phi_E}{dt}$ |
| Faraday's Law | $\sum_{curve} \vec{E} \bullet d\vec{s} = -\frac{\Delta \Phi_m}{\Delta t}$ | $\oint_c \vec{E} \bullet d\vec{s} = -\frac{d\Phi_m}{dt}$ |

$$\varepsilon_0 \frac{\Delta \Phi_E}{\Delta t} : \varepsilon_0 \frac{d\Phi_E}{dt}$$

is a correction term which Maxwell included because otherwise, Ampere's law is incomplete. You'll find out more about this also in advanced courses. For now, it is called the "displacement current."

Suppose you are in a charge and current free region of space. Then the last two equations tell you that a change in magnetic flux produces an electric field and a change in electric flux produces a magnetic field. We can show (although, the mathematics is a bit advanced) that this results in a transverse Electromagnetic wave which propagates with a velocity of

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3x10^8 \, m/s$$

This wave is described by an electric field vector which is perpendicular to a magnetic field vector. The wave propagates in a direction which is perpendicular to both E and B.

***(A TEM wave is a transverse electromagentic wave)***

Let's calculate the energy density of a TEM wave. The fields vary in time and space as

$$\begin{bmatrix} \vec{E} \\ \vec{B} \end{bmatrix} = \begin{bmatrix} E_m \\ B_m \end{bmatrix} \cos(kz - \omega t) \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}.$$

It turns out that the amplitudes $E_m$ and $B_m$ are also related by

$\frac{E_m}{B_m} = c$ and at each instant, $\frac{E}{B} = c$.

For a parallel plate capacitor, at one instant in time the electric energy density given by

$$u_E = \tfrac{1}{2} \varepsilon_0 E^2 .$$

We actually need the time average of this quantity which is at one particular point in space given by

$$\langle u_E \rangle = \tfrac{1}{2} \varepsilon_0 E_m^2 < \cos^2(\omega t) >= \frac{\varepsilon_0 E_m^2}{4}$$

Likewise the magnetic field energy density is given by

$$\langle u_B \rangle = \frac{B_m^2}{2\mu_0} < \cos^2(\omega t) >= \frac{B_m^2}{4\mu_0}$$

But, E and B are related by c and

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

The electric and magnetic parts contribute equally to the energy density. Thus

$$\langle u \rangle = \langle u_E \rangle + \langle u_m \rangle = \frac{\varepsilon_0 E_m^2}{2} = \frac{B_m^2}{2\mu_0} = \frac{\varepsilon_0 E_m^2}{4} + \frac{B_m^2}{4\mu_0}$$

**Poynting Vectors, intensity and other neat stuff**

The direction of energy transport in a TEM wave is given by the same direction as $\vec{E}\times\vec{B}$. In fact we can describe the rate of flow of energy in a TEM wave by a special vector called the ***Poynting Vector*** which is define by

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

The magnitude of the Poynting vector represents the rate at which energy flows through a unit surface area perpendicular to the flow. The SI units of S are [J/(sm$^2$)] or [W/m$^2$].

If we have a TEM wave, then there are lots of ways to write S. Let us make the following simplifications:

$\vec{E} = E_m\cos(kz-\omega t)\hat{x}$ is the Electric component of the TEM wave

$\vec{B} = B_m\cos(kz-\omega t)\hat{y}$ is the Magnetic component of the TEM wave

The instantaneous Poynting Vector is:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{E_m B_m}{\mu_0}\cos^2(kz-\omega t)\hat{z}$$

Note that I have used here the fact that $\hat{x}\times\hat{y}=\hat{z}$.

What is more useful, however is the time average Poynting vector. This is easy enough to calculate by now for my students because you know

$$\left\langle\cos^2(kz-\omega t)\right\rangle = \tfrac{1}{2}$$

So we can now easily determine

$$\left\langle\vec{S}\right\rangle = \frac{E_m B_m}{2\mu_0}\hat{z}$$

We have a special term for the time average Poynting vector: it is almost an ***intensity (I)***. In general, we defined an intensity as a power/unit area in the first semester ... here it is the same only we're really more interested in the time average power/unit area.

There are lots of ways to write <S>:

$$\left\langle\vec{S}\right\rangle = \frac{E_m B_m}{2\mu_0}\hat{z} = \frac{E_m^2}{2\mu_0 c}\hat{z} = \frac{cB_m^2}{2\mu_0}\hat{z}$$

and the term $\mu_0 c$ is called the ***impedance of free space***:

$$\mu_0 c = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\Omega$$

Hmm.. you might wonder why it's like a resistance. Here's a hand-waving argument:
S is like a power/area. Choose a region of space of area A with a normal parallel to S. Then the power passing through this area is SA. Now look at the second form of S:

$$\text{Power} = \frac{E_m^2 A}{2\mu_0 c}\ \text{which is like}\ \frac{\left\langle V^2\right\rangle}{Z} \Rightarrow Z = \mu_0 c$$

It is interesting (if not neat) to see how <S> is related to the time average energy density. We had earlier

$$u = u_E + u_M = \varepsilon_0 E^2 = \frac{B^2}{\mu_0}$$

and also $\left\langle u\right\rangle = \varepsilon_0\left\langle E^2\right\rangle = \tfrac{1}{2}\varepsilon_0 E_m^2 = \frac{B_m^2}{2\mu_0}$

It is immediately clear that the intensity and energy density are directly related by c:

$$I = \left\langle\vec{S}\right\rangle\bullet\hat{n} = c < u >$$

where $\hat{n}$ is the unit vector normal to an area which S is passing through.

Let's do an example now. Suppose a point source of electromagnetic radiation has an average power output of 800 W. Calculate the maximum values of the electric and magnetic fields at a point 3.50 m from the source.

Solution: For a point source of radiation (this also holds for distant stars and the like), the intensity which is defined as $I = \frac{\text{power}}{\text{area}}$ drops off as $1/r^2$. Why? Because you can consider that the power is spread equally over spheres of increasing radii. Thus, the surface area over which the power is spread is increasing as $\text{Area} = 4\pi R^2$. Thus, the intensity is

$$I = \frac{\langle \text{power} \rangle}{4\pi R^2}.$$

We also know that the intensity is related to the average energy density so

$$I = c\langle u \rangle = \frac{\langle \text{power} \rangle}{4\pi R^2}$$

We know what <u> is for a TEM wave and we are given what <power> is, so we can put all this together now to calculate $E_m$ and $B_m$: since

$$\langle u \rangle = \varepsilon_0 \langle E^2 \rangle = \tfrac{1}{2}\varepsilon_0 E_m^2 = \frac{B_m^2}{2\mu_0}$$

we then can see

$$\tfrac{1}{2} c\varepsilon_0 E_m^2 = \frac{<\text{power}>}{4\pi R^2}$$

which gives

$$E_m = \sqrt{\frac{2<\text{power}>}{4\pi R^2 c\varepsilon_0}} = \sqrt{\frac{2\bullet(800)}{4\pi\bullet(3.5)^2\bullet(3\text{x}10^8)\bullet(8.85\text{x}10^{-12})}} = 62.6\,\text{V/m}.$$

It is easiest now to calculate $B_m$ from the relation $\frac{E_m}{B_m} = c$ ('every body claps') ...

$$B_m = \frac{E_m}{c} = \frac{62.6}{3\text{x}10^8}\,\text{T} = 2.09\text{x}10^{-7}\,\text{T}$$

In a more advanced course, you'll find out that TEM waves are the only types of EM waves that can propagate in free space. Inside of cavities with cross sections approaching several wavelengths, these waves can not propagate (perhaps I can qualify this by saying "very successfully").

TEM waves transport momentum! (radiation pressure)

Recall the average energy density of a TEM wave:

$$\langle u \rangle = \varepsilon_0 \langle E^2 \rangle = \tfrac{1}{2}\varepsilon_0 E_m^2 = \frac{B_m^2}{2\mu_0}$$

Now, also think back to what an energy density is ... it is a pressure.

We obtained a pressure when we looked at the kinetic theory of an ideal gas:

$$\text{pressure} = \frac{\text{force}}{\text{area}} = \frac{\left(\frac{\Delta\text{momentum}}{\Delta\text{time}}\right)}{\text{area}}$$

Well let's equate this to the average energy density:

$$\langle u \rangle \bullet A \bullet \Delta t = \Delta p$$

Let's suppose we have a square of area A of the TEM wave which is propagating along. Then in a time $\Delta t$, the square will sweep out a volume given by A $c\Delta t$. Thus, we can use this to determine the total energy contained in the rectangular surface:

$$\langle U \rangle = \langle u \rangle Ac\Delta t$$

Thus we reach the conclusion that the momentum contained in the TEM wave is given by

$$\frac{U}{c} = p$$

so long as the TEM wave is completely absorbed by a surface ... if it bounces off of a surface, then there is a factor of 2 involved. Technically, I suppose you could say that this is due to a "re-radiation" from the reflecting surface ... the TEM wave does not really contain 2x the momentum just because it hits a reflecting surface.

Let's relate this to radiation pressure since someday you might want to use a solar sail to send a spaceship out of our solar system:

The energy is given by $U = pc$ so the rate of change of energy is $\frac{\Delta U}{\Delta t} = \text{power} = c\frac{\Delta p}{\Delta t}$.

But, $\frac{\Delta p}{\Delta t} = F$ (a force) and so the pressure is defined by F/A. But this is exactly the time average Poynting vector thus:

$$\frac{\langle S \rangle}{c} = P \quad \text{(P is the radiation pressure)}$$

If your sail is made of a completely black material the it delivers a momentum $p = \frac{U}{c}$ where as if your sail is completely reflecting, it delivers twice the momentum or

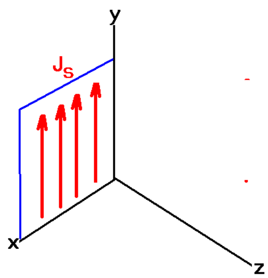
$$p = \frac{2U}{c}.$$

The radiation pressure on a completely reflecting surface is then $2\frac{\langle S \rangle}{c} = P$

Example: The sun delivers about 1000 $W/m^2$ of electromagnetic flux to the earth's surface. Calculate the total power that is incident on a roof of 200 $m^2$ and then determine the radiation pressure and the radiation force on the roof if it is a perfect reflector.

Solution: Power=<S>A=1000x200 = $2\times10^5$ W. Since the roof is completely reflecting, the radiation pressure is Pressure=2<S>/c=2000/$3\times10^8$=$6.67\times10^{-6}$ $N/m^2$. The total radiation force on the roof is then Force=Pressure x Area=$6.67\times10^{-6}$x200=$1.33\times10^{-3}$N.

Note: Lasers are capable of providing much higher values for <s> and in fact, it is possible to move objects with laser light (so called optical tweasers).




One final fantastic calculation to wrap it all up!

Suppose a conducting sheet is lying in the xy plane as shown. The sheet is carrying a surface current per unit length $J_s$. We will assume that $J_s=J_0\cos(\omega t)$. We want to be able to describe the TEM wave which propagates from this current sheet. I suppose it's appropriate to think of the plane as if it were an infinitely wide ribbon cable. Each wire in the cable carries a current I.

*Note: this is one of the few cases where we really get a true planar TEM wave. A wire, for example, produces a cylindrical TEM wave, in which case the intensity decreases with distance.*

Solution: we earlier have solved the problem of the infinite current sheet with Ampere's law. This gave the result that (you can verify this) $B_x = -\mu_0 \frac{J_s}{2}$. Here, since $J_s$ varies in time, this solution is really only going to be valid at points right next to the sheet. Thus, at the sheet of current we have the result $B_x = -\frac{\mu_0}{2} J_0 \cos(\omega t)$. This magnetic field has got to propagate in free space according to Maxwell's equations which say that it will be a TEM wave. Thus, as the magnetic field travels from the current sheet, it will obey:
$\vec{B}_x = -\frac{\mu_0}{2} J_0 \cos(kx-\omega t)\hat{x}$. The corresponding electric field will be easily related to the magnetic field: it must lie along the y axis in it's going to have a - sign since B has a - sign. Thus: $\vec{E}_y = -\frac{\mu_0 c}{2} J_0 \cos(kx-\omega t)\hat{y}$. The Poynting vector is then given by

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{\mu_0 c J_0^2}{4}\cos^2(kx-\omega t)\hat{z}.$$

In order to calculate the intensity of the TEM wave, we'll want to calculate <S>. Thus: $I = \langle \vec{S} \rangle \bullet \hat{n} = \frac{\mu_0 c J_0^2}{8}$. But, you need to remember that the current sheet is also producing a TEM wave propagating along the -z direction. This requires that the total rate of energy emitted per unit area of the conductor is $2\langle \vec{S} \rangle \bullet \hat{n}$.

Suppose the current sheet has a current density with a maximum value of 5 A/m (one way to think of this: you have, in 1 m, 5 wires, each carrying a current of 1A). Find the maximum values of the radiated magnetic and electric fields. Then, what is the maximum power incident on a second (completely absorbing) sheet parallel to the first sheet with an area of $3m^2$?

$$B_m = \frac{\mu_0 J_0}{2} \text{ and } E_m = \frac{\mu_0 J_0 c}{2}.$$

$$\text{So } B_m = \frac{4\pi x10^{-7} \bullet 5}{2} = 3.14x10^{-6}\text{T and so } E_m = 3x10^8 \bullet B_m = 942 V/m$$

The power is then the intensity x area or

$$\text{Power} = \frac{\mu_0 c j_0^2}{8} \bullet 3 = \frac{(4\pi x10^{-7})(3x10^8)(5^2)}{8} \bullet 3 = 3.54x10^3\,W$$