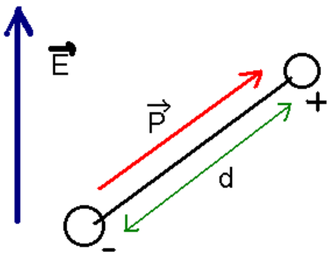
Dipole relaxation in a uniform electric field



Consider the electric dipole shown above:
In the presence of the electric field, the dipole experiences a torque given by:

$$\Gamma = 2\left(\tfrac{d}{2}\right) qE\sin\left(-\theta\right) = -qdE\sin\left(\theta\right) = -pE\sin\left(\theta\right)$$

The negative sign is because of the sense of measurement of the angle here.
For small angles, this torque becomes approximately:

$$\Gamma \approx -pE\theta$$

Since the time rate of change in angular momentum is proportional to the external torque, we have the equation of motion:

$$\tfrac{dL}{d\theta} = \tfrac{d(I\omega)}{dt} = I\tfrac{d^2\theta}{dt^2} = -pE\theta \Rightarrow \tfrac{d^2\theta}{dt^2} + \tfrac{pE}{md^2}\theta = 0$$

The solutions to this differential equation show simple harmonic oscillation with a frequency given by:

$$\omega_0 = \sqrt{\tfrac{pE}{md^2}}$$

Now what I want to do is to introduce a phenomenological dampening term and I want to drive with a sinusoidal electric field. The differential equation of motion which is then represented is a damped driven simple harmonic oscillator which has solutions that are well known in physics.

Here the differential equation is of the form:

$$\tfrac{d^2\theta}{dt^2} + b\omega_0\tfrac{d\theta}{dt} + \omega_0\theta = \tfrac{pE_0}{md^2}\sin\left(\omega t\right)$$

The solutions to this are well known:

$$\theta\left(t\right) = \tfrac{qE_0}{md^2\omega Z}\sin\left(\omega t + \varphi\right) : Z = \sqrt{\left(2\omega_0 b\right)^2 + \left(1 - \tfrac{\omega_0^2}{\omega^2}\right)^2} : \varphi = \tan^{-1}\left(\tfrac{2\omega\omega_0 b}{\omega^2 - \omega_0^2}\right)$$

Here the resonance frequency is given by:

$$\omega_r = \omega_0\sqrt{1 - 2b^2}$$

How is the connected to dielectric relaxation?
Let's look at the basis for dielectric relaxation: An external electric field induced an electric dipole moment in a sample (electric dipole moment is electric dipole per unit

volume). First, there are several types of charges which can occur in materials, free or bound. (For magnetism, we will also have a similar statement for currents). These charge densities may be either surface or volume. So there are at least 4 types of charges that we need to be concerned with. We define the polarization $\vec{P}$ of a material as the net dipole moment per unit volume arising from bound charges, regardless of how it happened. It may be permanent, or induced.

It is straight-forward to show by looking at the potential from a dipole:

$$V\left(\vec{r}_p\right) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}_{ip}\bullet\vec{p}}{\vec{r}_{ip}^{\,2}} = \frac{1}{4\pi\varepsilon_0}\int\limits_{\substack{\text{all}\\ q_i}} \frac{\hat{r}_{ip}\bullet\vec{P}}{\vec{r}_{ip}^{\,2}}\, d^3\vec{r}_i$$

That the contribution to the potential comes from two sources: volume bound charge densities and surface charge densities. We show this by use of (as your author does) the result:

$$\vec{\nabla}_i\left(\frac{1}{r_{ip}}\right) = \frac{\hat{r}_{ip}}{\vec{r}_{ip}^{\,2}}$$

For the potential above, this gives:

$$V\left(\vec{r}_p\right) = \frac{1}{4\pi\varepsilon_0}\int\limits_{\substack{\text{all}\\ q_i}} \vec{P}\bullet\vec{\nabla}_i\left(\frac{1}{r_{ip}}\right) d^3\vec{r}_i$$

Following your author, this can be integrated by parts:

$$V\left(\vec{r}_p\right) = \frac{1}{4\pi\varepsilon_0}\int\limits_{\substack{\text{all}\\ q_i}} \vec{P}\bullet\vec{\nabla}_i\left(\frac{1}{r_{ip}}\right) d^3\vec{r}_i = \frac{1}{4\pi\varepsilon_0}\left[\int\limits_{q_i}\vec{\nabla}_i\left(\frac{\vec{P}}{r_{ip}}\right)d^3 r_i - \int\limits_{q_i}\frac{1}{r_i}\left(\vec{\nabla}_i\bullet\vec{P}\right)d^3 r_i\right]$$

Now the divergence theorem says that we may write the first part as a surface integral and the second as a volume integral (still in reference to the volumes containing the charges):

$$V\left(\vec{r}_p\right) = \frac{1}{4\pi\varepsilon_0}\left[\int\limits_{s_i}\frac{\vec{P}\bullet d\vec{A}_i}{r_{ip}} - \int\limits_{v_i}\frac{1}{r_{ip}}\left(\vec{\nabla}_i\bullet\vec{P}\right)d^3 r_{ip}\right]$$

The first term corresponds to an effective contribution to the potential from a bound surface charge density:

$$\sigma_b = \vec{P}\bullet\hat{n}$$

And the second term corresponds to an effective contribution to the potential from a bound volume charge density:

$$\rho_b = -\vec{\nabla}\bullet\vec{P}$$

That these are indeed bound is due to the fact that nowhere has anything been said about an external field in this derivation.

It is worth noting that this derivation used the pure dipole potential which may well not apply within materials. We move on though.

Now what we want is to find the electric field which arises from all charges, bound or free. Basically is the charges are not bound, we're going to call them free. In general, the differential form of Gauss's law says:

$$\vec{\nabla}\bullet\vec{E} = \frac{\rho}{\varepsilon_0}$$

Now however the charge density is due to free and bound charges:

$$\rho = \rho_f + \rho_b$$

So we rewrite Gauss's law as:

$$\varepsilon_0\vec{\nabla}\bullet\vec{E} = \rho_b + \rho_f = -\vec{\nabla}\bullet\vec{P} + \rho_f \Rightarrow \vec{\nabla}\bullet\left(\varepsilon_0\vec{E} + \vec{P}\right) = \rho_f$$

We thus can identify the portion of the electric field type entity that arises purely from the free charges as:

$$\vec{D} \equiv \varepsilon_0\vec{E} + \vec{P}.$$

This is called the electric displacement and in terms of the displacement, Gauss's law is:

$$\vec{\nabla}\bullet\vec{D} = \rho_f$$

So before we go further, what is D? Is it safe to say that D is basically that part of the electric field that arises solely from free charges?

Points: your author says there is no coulomb's law for D. In particular

$$D \neq \frac{1}{4\pi}\int \frac{\rho_f(r_i)}{r_{ip}}\hat{r}_{ip}d^3r_i$$

Note though: these two statements are equivalent:

$$\vec{\nabla}\bullet\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}\bullet\vec{D} = \rho_f$$

It is obvious but seems important there to point out that D and E have different units: since the permittivity of free space is $\varepsilon_0 = 8.85\text{x}10^{-12}\frac{C^2}{Nm^2}$, and E is defined as having SI units of N/C, we thus have the units of D to be C/m^2. Notice that, however P and D have the same units and this might suggest thinking of D more in terms of a polarization resulting from free charges as compared to P resulting from bound charges. So clearly they are not even close to being the same. More rightly, D even seems to be more of a surface density of charge based upon its units and indeed, looking at the divergence of this gives us a volume charge density. However, to interpret it in this manner ignores one important point: D is a vector while surface charge density is a scalar.

Between the plates of an ideal parallel plate capacitor, the **magnitude** of D would be equal to the surface charge density on a plate. If a linear dielectric material is inserted into the capacitor, the material polarizes with the result that additional charge must flow into the plates in order to maintain the same potential difference between the plates. Then, the value of the **magnitude** of D must increase because there are now more free charges on the plates. The constant potential difference can be maintained by a battery for example.

Let's see how things work: for a linear dielectric, the polarization is given by:

$$\vec{P} = \varepsilon_0\chi_e\vec{E}$$

Where χ_e is called the electric susceptibility.

We also have for D:

$$\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \left(1 + \chi_e\right) \vec{E} = \varepsilon \vec{E}$$

The permittivity **of the material** is thus defined by:

$$\varepsilon = \varepsilon_0 \left(1 + \chi_e\right)$$

The dielectric constant or relative permittivity is now defined by:

$$\varepsilon_r \equiv \tfrac{\varepsilon}{\varepsilon_0} = \left(1 + \chi_e\right)$$

So we want to calculate all imaginable quantities now inside the capacitor.

Note: **D, E and P are all in the same direction**.

Now remember the dipole moment of the material: the vector dipole moment points from – charges towards + charges of the bound type. **This means that** if the direction of the dipole moment is not defined this way, that there is a sign problem for linear dielectrics. Also please note that the polarity of the dielectric has negative bound charges close to the plate with positive charges. This is important to keep in mind.

When the dielectric is inserted in between the plates of a capacitor, an amount of surface charge σ_{new} will be on the plates. From this we can calculate D:

$$\vec{D} = \sigma_{new} \hat{x}$$

We can now calculate E inside the material:

$$\vec{E} = \tfrac{\vec{D}}{\varepsilon} = \tfrac{\sigma_{new}}{\varepsilon} \hat{x}$$

The polarization is then given by:

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 \chi_e \tfrac{\sigma_{new}}{\varepsilon} \hat{x} = \varepsilon_r \chi_e \sigma_{new} \hat{x}$$

The bound charge density next to the positive plate is given by:

$$\sigma_b = \vec{P} \bullet \hat{n} = -\varepsilon_r \chi_e \sigma_{new}$$

We can find the potential difference by integrating E:

$$\Delta V = -\int_0^a \vec{E} \bullet d\vec{l} = -Ea$$

So if we want the potential difference between the old and new systems to be the same, we can determine how much charge went onto the plate:

$$\tfrac{\sigma_{new}}{\varepsilon} = \tfrac{\sigma_{old}}{\varepsilon_0} \Rightarrow \sigma_{new} = \sigma_{old} \tfrac{\varepsilon}{\varepsilon_0} = \varepsilon_r \sigma_{old}$$

Which gives an idea of a method for measurement of polarization, and all the quantities: Take a battery and connect it between the plates of a parallel plate capacitor. In the circuit connect an electrometer (or a very sensitive ammeter). Insert the dielectric material between the plates of the material so that the current flow is constant, recording the full time the current vs time that flows onto the plates. Then you can find the new charge on the capacitor by:

$$\sigma_{new} = \sigma_{old} + I\Delta t$$

You can now find the dielectric constant by:

$$\tfrac{\sigma_{new}}{\sigma_{old}} = \varepsilon_r = 1 + \tfrac{I\Delta t}{\sigma_{old}} = 1 + \tfrac{I\Delta t}{\varepsilon_0 E_{old}} = 1 + \tfrac{I\Delta t}{\left(\frac{\varepsilon_0 E_{old} w}{w}\right)} = 1 + \tfrac{wI\Delta t}{\varepsilon_0 (\Delta V)}$$

Although for sure you would not exactly want to do it this way.

Once you have ε_r, you can find χ_e by $\chi_e = \varepsilon_r - 1 = \tfrac{wI\Delta t}{\varepsilon_0 (\Delta V)}$

A simpler thought experiment though is to measure C and $C_{geometrical}$ and then the dielectric constant is given by:

$$\varepsilon_r = \frac{C}{C_{geometrical}}$$

For a parallel plate capacitor, the geometrical capacitance is given by:

$$C_{geo} = \varepsilon_0 \frac{Area}{w}$$

And one technique is to measure the capacitance by the following:

Using the definition of capacitance:

$$C = \frac{Q}{V} \Rightarrow Q = CV \Rightarrow I = C\frac{dV}{dt}$$

Do the following: increase the potential at a constant rate so that $\frac{dV}{dt} = b$.

Then measure at the same time the current going into the capacitor:

The result then upon dividing these two measurements is:

$$C = \frac{I}{\left(\frac{dV}{dt}\right)} = \frac{I}{b}$$

In fact, this can be used to calibrate electrometers with known capacitances.

It is important to note several things, one of which is that the line integral of the curl of P does not vanish on a path that lies partially outside the dielectric material. This means that the differential connections between E, D and V are significantly more complicated than just replacing E by D with a constant. Notice you are saved from some of these problems by putting electrodes (thick if necessary) on your dielectric sides. Also note that in general the polarization is not uniform in all three directions for most materials and indeed may have nine different electric susceptibilities depending upon the material. These comprise the susceptibility tensor.

We calculate the energy density for a **linear** dielectric as:

$$u = \frac{1}{2}\varepsilon_0\varepsilon_r E^2 = \frac{1}{2}\vec{D}\bullet\vec{E}$$

However this **explicitly** does not apply necessarily to electric materials with a permanent polarization (ferroelectrics, for example).

However for other systems, you need to directly go to the calculation of E as

$$u = \frac{1}{2}\varepsilon_0\vec{E}^2$$

It would seem then that understanding energy storage in ferroelectrics might be of some interest.