Section 2.5

Before we start looking at solutions to Laplace's equation and Poisson's equation, we need to investigate boundary conditions in conductors, because it is the boundary conditions that provides us with the uniqueness to the fields that arise from potential solutions.

what happens to a conductor in the presence of an electric field?

Firstly, let's define a conductor as an equipotential surface. This means that no work is required to move a charge on such a surface.

If no work is required to move a charge on such a surface, there can be no electric field within a conductor. We're going to have to account for this somehow.

Here is how: the presence of an external field impinging upon a conductor will polarize the conductor and produce an internal field which is exactly opposite and equal in magnitude to the external field acting upon the conductor. The superposition of these two fields then produces no net internal field in a conductor.

## properties of a conductor

(1) E =0 inside a conductor. Of this, you can rest assured, so long as we stick with electrostatics. If E were not zero, charges would flow inside the conductor, thus making E zero. If the flowing charges did not ultimately make any electric field inside the conductor zero, you'd have a violation of energy conservation.

(2) $\rho$ =0 inside a conductor. This is a straight-forward consequence of (1) in that

$$\vec{E} = \vec{0} \Rightarrow \vec{\nabla} \bullet \vec{E} = 0 \Rightarrow \rho = 0$$

(3) Some conductors carry a charge and it can not be a volume charge. It therefore must be a surface charge $\sigma$. As your author says, that's the only place left for charge to be.

(4) By our definition of a conductor, it is an equipotential surface which means:

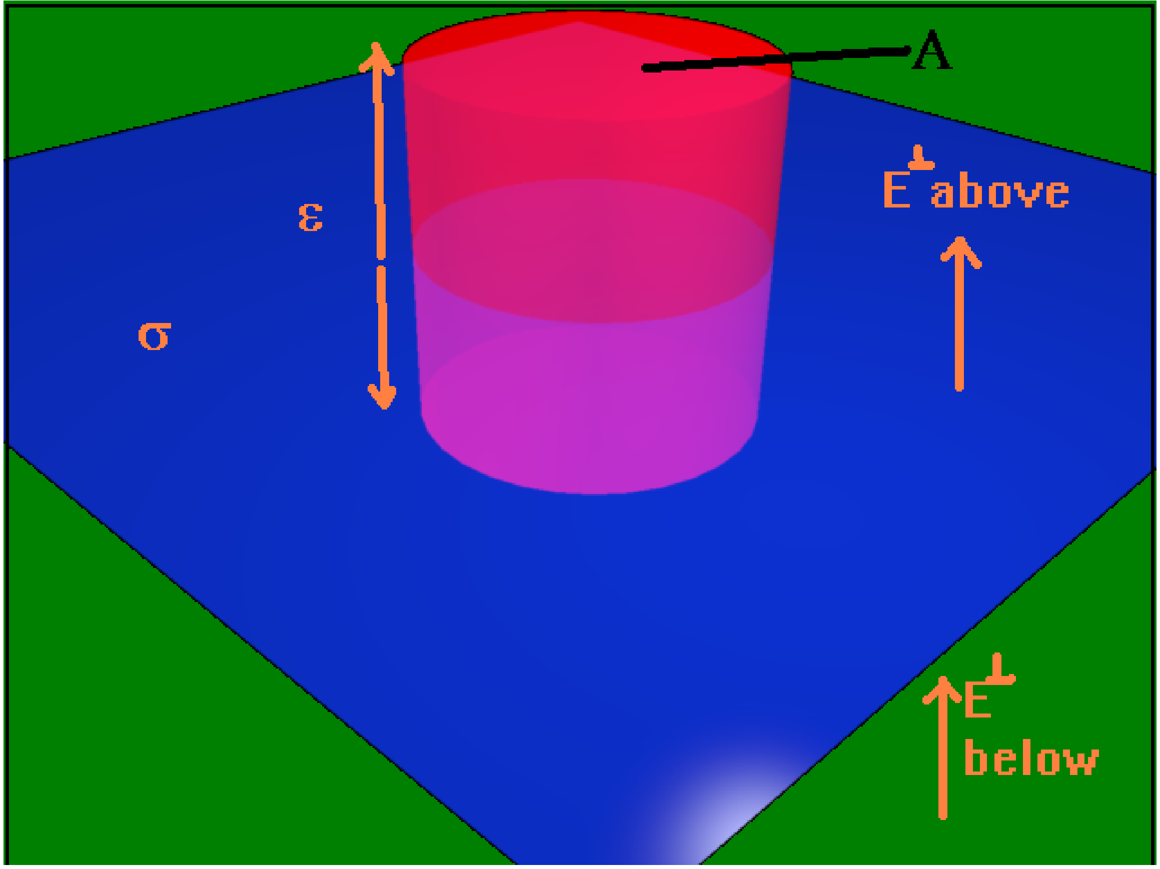
$$\Delta V = -\int_a^b \vec{E} \bullet d\vec{l} = 0 \,.$$

(5) E does exist outside of conductors, and just outside a conductor, any E must be normal to the surface. If there were an E tangential to the surface, the surface charges would flow to cancel out the tangential components of E.

Section 2.5.2: Induced Charges

Please read this section closely.

Let's obtain the boundary condition for a boundary. (this refers to section 2.3.5)



Gauss's law applied to my pretty picture above shows:

$$\oint_{\text{surface}} \vec{E} \bullet d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

We can also evaluate the change in E across this boundary:

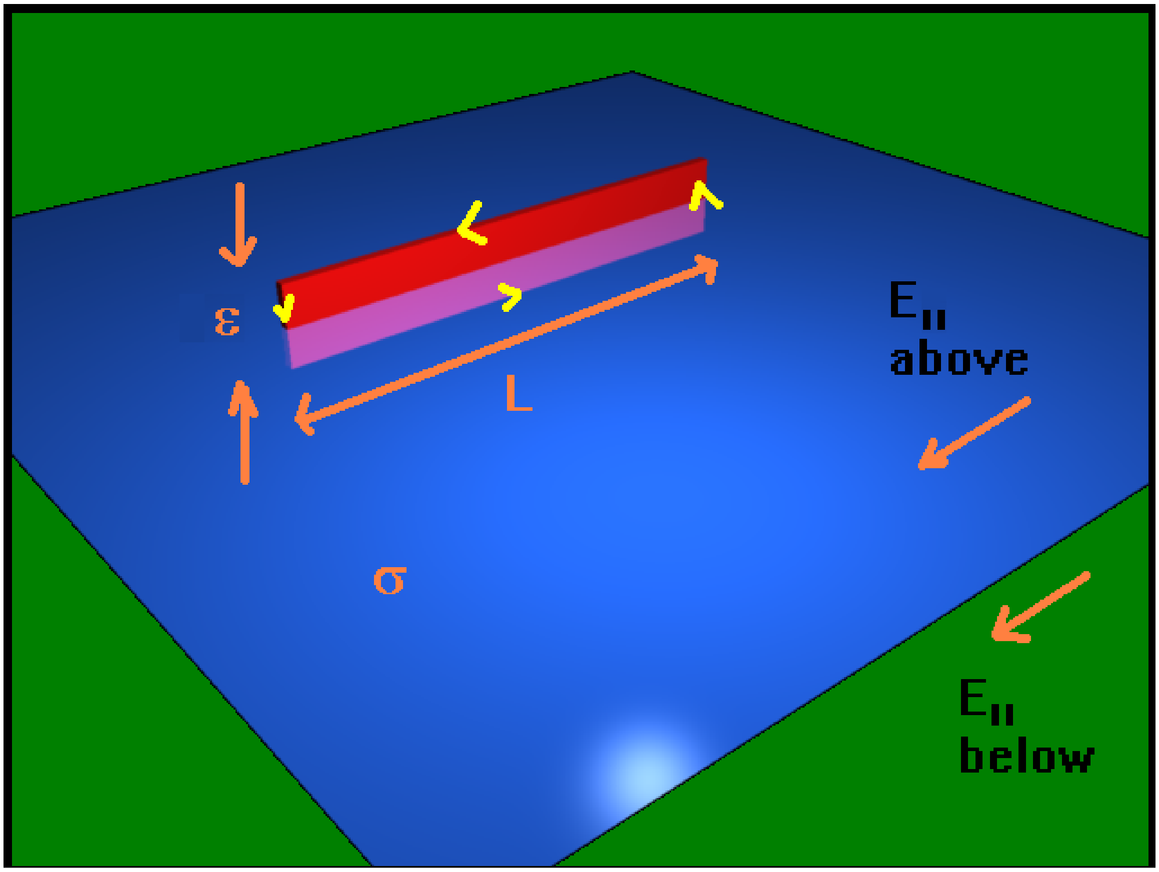
$$\oint_{\text{surface}} \vec{E} \bullet d\vec{A} = A\left(E_n^{above} - E_n^{below}\right)$$

Thus there is a discontinuity in the normal component of E which is given by:

$$\left(\vec{E}^{above} - \vec{E}^{below}\right) \bullet \hat{n} = \frac{\sigma}{\varepsilon_0}$$

The tangential component of E, on the other hand, is continious:

$$\left(\vec{E}^{\text{above}} - \vec{E}^{\text{below}}\right) \times \hat{n} = \vec{0}$$



$$\oint_{\text{Path}} \vec{E} \bullet d\vec{L} = 0$$

The ends do not contribute anything since we'll let ε->0. The only possible conclusion, then that can be reached from this is:

$$\vec{E}^{\text{above}} \times \hat{n} = \vec{E}^{\text{below}} \times \hat{n}$$

or, the tangential component of E is continuous.

We also have then that the potential V across the surface must be continuous also since:

$$\Delta V = \lim_{\varepsilon \to 0} \left[ -\int_{-\frac{\varepsilon}{2}}^{+\frac{\varepsilon}{2}} \vec{E} \bullet d\vec{L} \right] = 0 \Rightarrow V^{\text{above}} = V^{\text{below}}$$

The gradient of V is, however, discontinuous:

$$\vec{E} = -\vec{\nabla} V \Rightarrow \vec{\nabla} V^{\text{above}} - \nabla \vec{V}^{\text{below}} = -\tfrac{\sigma}{\varepsilon_0}$$

If we define the normal derivative by:

$$\tfrac{\partial V}{\partial n} = \vec{\nabla} V \bullet \hat{n}$$

Then, this can be written as:

$$\frac{\partial V^{\text{above}}}{\partial n} - \frac{\partial V^{\text{below}}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

**In the rush to go onto other things, it is easy to miss section 2.5.3:**

We shouldn't however since it'll explain all kinds of practical things that you'll often see in the lab.

I'll follow your text closely here.

Let's look at the surface charge and forces on a conductor.

Let's place a conductor into an electric field. Immediately outside the conductor, the electric field is given by:

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

This is also expressible in terms of the potential as:

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$$

Let's find the electric field in the region of a small patch, as your author does.

The electric field in the region of the small patch is

$$\vec{E} = \vec{E}_{patch} + \vec{E}_{other}$$

Now, if the patch carries a charge $\sigma$, then the electric fields are:

$$\vec{E}_{above} = \vec{E}_{other} + \frac{\sigma}{2\varepsilon_0}\hat{n}$$

$$\vec{E}_{below} = \vec{E}_{other} - \frac{\sigma}{2\varepsilon_0}\hat{n}$$

Add these two expressions to obtain:

$$\vec{E}_{other} = \frac{1}{2}\left(\vec{E}_{above} + \vec{E}_{below}\right) = \vec{E}_{average}$$

This is the electric field which will be acting on the patch, trying to make it do things it normally might not want to do.

Well, if the patch were a conductor, so that below there were no electric field, then:

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

If the path then has a surface charge density $\sigma$, we obtain a force on the conductor:

$$\vec{f}\,/\,\text{area} = \frac{\sigma^2}{2\varepsilon_9}\hat{n}$$

This is, of course, an electrostatic pressure. In terms of the electric field just outside the conductor:

$$P = \frac{1}{2}\varepsilon_0 E^2$$

You'll recall this is also the energy density for a parallel plate capacitor!

In any event, this is an outward electrostatic pressure on the surface tending to draw the conductor into the field, regardless of the sign of the surface charge density. If you're going to be manipulating nanites around, you might just find such a property useful.

Hmm … can you imagine uncompressing a block of copper?

### 2.5.4 Capacitors

(geometrical capacitance)

We define capacitance as the relative efficiency for a given potential to store charge. The definition of capacitance can be thought of as the charge analogue to Ohm's law.

$$C \equiv \frac{Q}{V}$$

Here, Q is the charge separation between two electrodes (plates) and V is the potential difference between the two plates. In fact, I think the definition is most reliable when applied to conducting surfaces.

It is easy to determine the capacitances for several simple geometries:

(a) Consider a solid conducting sphere of radius a.

$$\oint \vec{E} \bullet d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \Rightarrow V = k\frac{Q}{r} \Rightarrow \Delta V = k\frac{Q}{a}$$

It is easy to see from this then that the capacitance of a sphere is:

$$C = \frac{a}{k} = 4\pi\varepsilon_0 a$$

For the parallel plate capacitor, we had a uniform electric field between the plates. So

$$\Delta V = Ed = \frac{\sigma}{\varepsilon_0} d$$

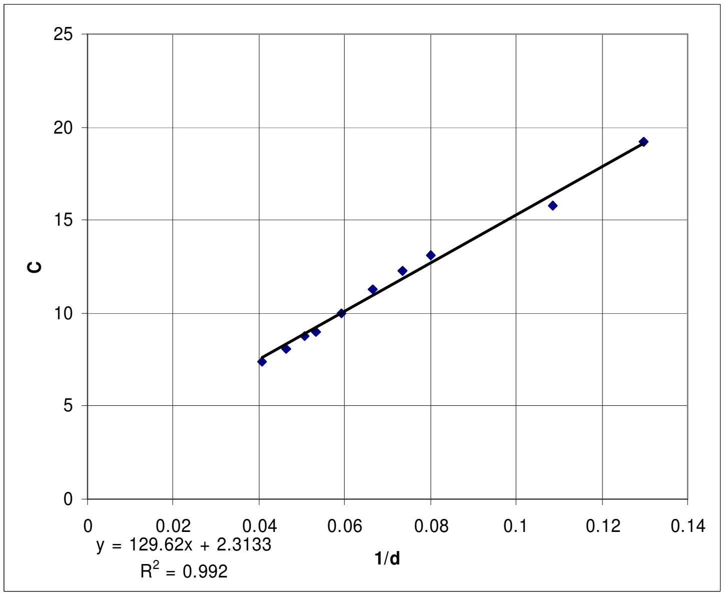
The total charge separation is related to the area of the plates:

$$Q = \sigma A$$

It is easy from this to see that the capacitance of the parallel plate capacitor is:

$$C = \varepsilon_0 \frac{A}{d}$$

The 1/d behavior is pretty easy to verify with the two copper plates. Here is a plot from just such an experiment.



Another important and easy geometry for capacitance calculations is the coaxial cable:

The inner conductor has a radius a and the outer conductor has a radius b. The inner conductor carries +λ while the outer conductor caries -λ. For a Gaussian surface of height h, we have:

$$\oint \vec{E} \bullet d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow E\left[2\pi s h\right] = \frac{\lambda h}{\varepsilon_0} \Rightarrow \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s} \Rightarrow \Delta V = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

From this we can calculate the capacitance of the coaxial cable:

$$C = \frac{Q}{V} = \frac{\lambda h}{\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)}h$$

Often, however, this is specified as the capacitance per unit length:

$$\frac{C}{h} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)}$$

Here is yet another example: consider two concentric conducting spheres of radius a and b. The inner sphere carries +Q and the outer sphere carries -Q. Calculate the capacitance.

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} : \Delta V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right] \Rightarrow C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]} = \frac{4\pi\varepsilon_0}{\left[\frac{1}{a} - \frac{1}{b}\right]}$$

Here is a new way to calculate the stored energy in a capacitor.

Move the first charge across the plates. no work is done.
Move the second charge across the plates. The work done is:

$$w_1 = q\left[\frac{q}{C}\right]$$

move the third charge across the plates. The work done is:

$$w_2 = q\left[\frac{2q}{C}\right]$$

...

move the nth charge across the plates. The work done is:

$$w_n = q\frac{nq}{C}$$

Add up all the works:

$$U = \frac{q^2}{C}\left[1 + 2 + 3 + ... + n\right] = \frac{q^2}{C}\sum_{i=1}^{n} i = \frac{q^2}{C}\left(\frac{n(n+1)}{2}\right) \approx \frac{q^2}{2C}\left(n^2\right) = \frac{(nq)^2}{2C} = \frac{Q^2}{2C}$$

So the energy stored in a capacitor is:

$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

Done the calculus way:

$$U = \int_{q=0}^{q=Q} V dq = \int_{q=0}^{q=Q} \frac{q}{C} dq = \frac{Q^2}{2C}$$