Two Important Laws of Electrostatics
Disclaimer: be careful with electricity
http://books.google.com/books?vid=0HVxDeWcFJrO-iS0&id=DBoAAAAAQAAJ&pg=PA151&lpg=PA149&dq=electric+++date:1700-1770&as_brr=1#PRA5-PA150,M1

(see paragraph 319 regarding Professor Richmann)
Also see page 299 (text numbered page) Death report on page 159
Many electrostatics experiments can not be done except in the winter. Here I'll describe some of the experiments that show particular things of importance.

Law of Charges:

Like charges attract. Unlike charges repel.

Experiments with the law of charges show that there are 2 types of charge: + and -.

To produce a negative charge: the modern day definition is simply that a negative charge is that charge which is developed on a rubber rod when it is rubbed with cat's fur.

How can we know from this definition that there are two types of charge?

Experiment 1:

Rub a rubber rod with cat's fur.
Touch the rod to a small piece of Styrofoam which is attached to a string.
The ball will immediately be repelled away from the rod. Since the rod had a negative charge, the ball also has a negative charge.

Experiment 2:

Rub a glass rod with a piece of acetate cloth.
Bring it near the pith ball. The ball will be attracted to the glass rod. Since the ball had a negative charge, and it was attracted to the glass rod, while being repelled from the rubber rod, the only possible conclusion is that the glass rod has the opposite sign (+).

Coulomb's Law

Coulomb's law is the mathematical quantification of the observations of the response of one charge to another charge. Mathematically, it is stated as:

$$\vec{F}_{\text{on 1 due to 2}} = k\frac{q_1 q_2}{|\vec{r}_{12}|^2}\hat{r}_{12}$$

Which says this: the force on charge 1 due to charge 2 is
(1) proportional to the inverse distance between the two charges squared
and
(2) points along the line separating the two charges.

These two laws, actually pretty well sum up electrostatics so we might as well stop at this point and say that things are pretty well complete. However, what is done with these two laws is remarkable. Also notice, that since the sign of the charge is present in Coulomb's law, that we really have 1 single law to describe all of electrostatics!

In my physics classes, we spend time calculating each of the vectors and the distances. This is, in fact, one of the hardest parts for students. In this workshop, I will show you how to do this, and we will do it for a few simple examples. I will provide a spreadsheet that will let you calculate the force on a charge for more complicated charge distributions.

A really nice thing to remember is what is the value of Coulomb's constant. It is easiest to remember it as: $k\mu = 8990\,\mathrm{N}\frac{m^2}{C^2}$

so $k\mu\mu = 8990\,\mu\mathrm{N}\frac{m^2}{C^2}$

since realistic quantities of charges are expressed in micro-(μ) Coulombs or 10^{-6}C.

Calculations of electrostatic force from charge distributions

A first example: (the electric dipole)

#1:(q=+1μC,x=5m,y=1m):#2:(q=-1μC,x=4m,y=2m)

Calculate the Vector force

(1) Calculate the vector $\vec{r}_{12}$

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1 = (4\hat{x} + 2\hat{y}) - (5\hat{x} + 1\hat{y}) = (4-5)\hat{x} + (2-1)\hat{y} = -1\hat{x} + 1\hat{y}$$

This vector points 1 unit in the –x direction and up 1 unit in the y direction.

(2) Calculate the magnitude of this vector

This comes from Pythagorean's theorem:

$$|\vec{r}_{12}| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

(3) Calculate the unit vector.

$$\hat{r}_{12} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|} = \frac{-1\hat{x}+1\hat{y}}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y}$$

Now put everything together to calculate the force:

$$\vec{F}_{12} = 8990\mu\frac{(+1)(-1)}{[\sqrt{2}]^2}\left[-\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y}\right]\mathrm{N} = \frac{8990\mu}{[\sqrt{2}]^2}(\hat{x} - \hat{y}) = 3178\mu(\hat{x} - \hat{y})\mathrm{N}$$

This is a really big force. How big? To correctly answer this you ought to find the magnitude of this force:

$$|\vec{F}_{12}| = \frac{3178}{\sqrt{2}} = 2248\mu\mathrm{N}$$

It's not actually that big of a force: it is about 0.002N.

Now that you know how to do a simple calculation with electrostatic force, the question arises: what do you do if you have 3 charges. The ultimate answer is add up all the forces Vectorially. Mathematically, this looks like:

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13}$$

In general, you would write this as:

$$\vec{F}_1 = \sum_{i=2}^{N} \vec{F}_{1,i}$$

Here is a spreadsheet which will do this for up to 10 charges
http://www.lyon.edu/webdata/users/shutton/Courses/Su08/WorkShop/ ElectrostaticForceCalculation.xls
This calculation looks tedious ... it is. However, it is also important for students to understand these basics before progressing further.

We do not always want, however the electrostatic force when working with electrostatics.

The Electric Field

A very good applet that we will use over and over again is:
http://www.falstad.com/vector3de/

We actually want to use the Electric Field.
The electric field is defined as the net electrostatic force per unit positive test charge.
It points in the direction that a positive charge would accelerate (and move) if released from rest. If the charge is already moving, it would only be the acceleration of the positive charge.

Mathematically: $\vec{E}_p = \frac{\vec{F}}{q_+} = \sum_{i=1}^{n} k \frac{q_i}{|\vec{r}_{ip}|^2}\hat{r}_{ip}$

This tells you how to calculate the electric field at a point p in space due to several charges (n charges) which are distributed throughout the region of interest.
In general, this is a pretty tedious calculation but is also straightforward if you apply the rules for vectors.

Let me show you an example.
Example 2:

A charge –q is located at x=-1 and a charge +q is located at x=+1. Find the vector electric field at a point y_p along the positive y-axis.

You need to apply the definition of the electric field:

$$\vec{E}_p = \sum_{i=1}^{2} k \frac{q_i}{|\vec{r}_{ip}|^2}\hat{r}_{ip}$$

Now to do this, you will need to calculate each of the vectors.

(I'll make a sketch on the board here)

It is best to list everything you have for this problem: (lengths in m here)
(#1:-q,x=-1,y=0)
(#2:+q,x=+1,y=0)

(point, x_p=0,y=y_p)

Now you want to calculate the vectors:

$$\vec{r}_{1p} = \vec{r}_p - \vec{r}_1 = \left(0\hat{x} + y_p\hat{y}\right) - \left(-1\hat{x} + 0\hat{y}\right) = 1\hat{x} + y_p\hat{y}$$

$$\vec{r}_{1p} = \vec{r}_p - \vec{r}_1 = \left(0\hat{x} + y_p\hat{y}\right) - \left(1\hat{x} + 0\hat{y}\right) = -1\hat{x} + y_p\hat{y}$$

Now calculate the magnitudes:

$$\left|\vec{r}_{1p}\right| = \sqrt{1^2 + y_p^2} : \left|\vec{r}_{2p}\right| = \sqrt{1^2 + y_p^2}$$

Notice the magnitudes in this case are the same. That will make life a bit easier.

Now calculate the unit vectors.

$$\hat{r}_{1p} = \frac{1\hat{x} + y_p\hat{y}}{\sqrt{1^2 + y_p^2}} : \hat{r}_{2p} = \frac{-1\hat{x} + y_p\hat{y}}{\sqrt{1^2 + y_p^2}}$$

Now you are ready to put it all together:

$$\vec{E}_p = \vec{E}_{1p} + \vec{E}_{2p} = k\frac{q_1}{\left[\sqrt{1^2 + y_p^2}\right]^2}\frac{1\hat{x} + y_p\hat{y}}{\sqrt{1^2 + y_p^2}} + k\frac{q_2}{\left[\sqrt{1^2 + y_p^2}\right]^2}\frac{-1\hat{x} + y_p\hat{y}}{\sqrt{1^2 + y_p^2}}$$

Now you will want to simplify this expression:

$$\vec{E}_p = k\frac{1}{\left[\sqrt{1^2 + y_p^2}\right]^3}\left(-q\left(1\hat{x} + y_p\hat{y}\right) + q\left(-1\hat{x} + y_p\hat{y}\right)\right) = k\frac{q}{\left[\sqrt{1^2 + y_p^2}\right]^3}\left(-2\hat{x}\right)$$

As we get very far from the dipole, the electric field will drop off as

$$\vec{E}_p = -2k\frac{q}{y_p^3}\hat{x}$$

Which has always amazed me: the electric field from a single point charge drops off as $1/r^2$ but if you add up the field of an opposite charge, along the symmetry axis it drops off as $1/r^3$. In the general physics labs, we actually make a measurement with this to show this to be true.

Provided there is enough time, we can do this pretty simply in today's lab.

I have as a class activity now to calculate the electric field at any point in the positive y region, not just along the symmetry axis. The modification is this:

(#1:-q,x=-1,y=0)

(#2:+q,x=+1,y=0)

(point, x=x_p,y=y_p)

Solution:

Vectors look like this:

$$\vec{r}_{1p} = \vec{r}_p - \vec{r}_1 = \left(x_p\hat{x} + y_p\hat{y}\right) - \left(-1\hat{x} + 0\hat{y}\right) = \left(1 + x_p\right)\hat{x} + y_p\hat{y}$$

$$\vec{r}_{1p} = \vec{r}_p - \vec{r}_1 = \left(x_p\hat{x} + y_p\hat{y}\right) - \left(1\hat{x} + 0\hat{y}\right) = \left(-1 + x_p\right)\hat{x} + y_p\hat{y}$$

Magnitudes look like this:

$$\left|\vec{r}_{1p}\right| = \sqrt{\left(1 + x_p\right)^2 + y_p^2} : \left|\vec{r}_{2p}\right| = \sqrt{\left(-1 + x_p\right)^2 + y_p^2}$$

Unit vectors look like this:

$$\hat{r}_{1p} = \frac{\left(1+x_p\right)\hat{x}+y_p\hat{y}}{\sqrt{\left(1+x_p\right)^2+y_p^2}} : \hat{r}_{2p} = \frac{\left(-1+x_p\right)\hat{x}+y_p\hat{y}}{\sqrt{\left(-1+x_p\right)^2+y_p^2}}$$

The electric field looks like this:

$$\vec{E}_p = \vec{E}_{1p} + \vec{E}_{2p} = k\frac{-q}{\left[\sqrt{\left(1+x_p\right)^2+y_p^2}\right]^2}\frac{\left(1+x_p\right)\hat{x}+y_p\hat{y}}{\sqrt{\left(1+x_p\right)^2+y_p^2}} + k\frac{+q}{\left[\sqrt{\left(-1+x_p\right)^2+y_p^2}\right]^2}\frac{\left(-1+x_p\right)\hat{x}+y_p\hat{y}}{\sqrt{\left(-1+x_p\right)^2+y_p^2}}$$

This does not immediately simplify a whole lot ... it can be expressed more simply but that takes us beyond what we want to do here. This can also be calculated in an excel spreadsheet point by point.

http://www.lyon.edu/webdata/users/shutton/Courses/Su08/WorkShop/ElectrostaticFieldDipoleCalculation.xls

I like this applet also for sketching electric field lines:

http://www.cco.caltech.edu/~phys1/java/phys1/EField/EField.html

Several important things to come away from this with so far:
What is an electric field?
How is it calculated?
Which way does it point?

Note: If I have not said so by now, electric field vectors point away from negative charges and towards positive charges.

Now I want to teach you how to draw electric field lines of force. The rules you need here are these:
The lines are perpendicular to the surface of a conductor.
The lines do not cross.
The lines terminate on positive charges and come from negative charges.
The more line density you have, the greater the electric field is in that region of space.

Sketching Electric Field Lines

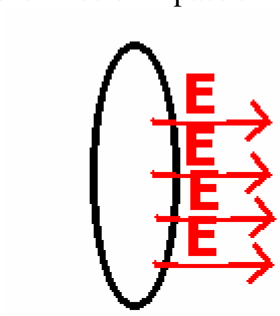
Now let's draw the electric field lines for a point charge.
Stretch the point charge into a plane, and then an infinite plane
Now lets draw the electric field lines for a dipole
Stretch the dipole into 2 planes, and then 2 infinite planes.

In principle all electric fields from charge distributions can be calculated in this way. In practice, you really need more advanced techniques for charged structures, which is why I am going to introduce you to Gauss's Law.

Gauss's law

Because of time limitations, we're going to cut to the chase here to show you how Gauss's law works. There are a limited number of applications that is can be easily applied to.

(a) What is electric flux?
Suppose that the lines of E pass through a surface.



If the lines pass normally through the surface (you may need me to explain what I mean by normally) then the electric flux is given by:

$$\Phi_E = E(\text{Area})$$

If the electric field passes at some angle to the surface, then the electric flux is given by

$$\Phi_E = E(\text{Area})\cos(\theta)$$

where the angle theta is the angle between the normal to the surface and the electric field.

Now an important mathematical theorem (Gauss's Law) says that if the surface is simply closed (meaning it does not have any holes in it), then

$$\Phi_E = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

where $k = \frac{1}{4\pi\varepsilon_0}$

I will sketch this on the board for you.
Here is an example for a point charge +Q. Choose a sphere of radius r centered on the charge. The electric flux is then

$$\Phi_E = E(\text{Area}) = E(4\pi r^2)$$

The total charge enclosed is Q so we then apply Gauss's Law:

$$E(4\pi r^2) = \frac{Q}{\varepsilon_0}$$

Then solve this for the electric field:

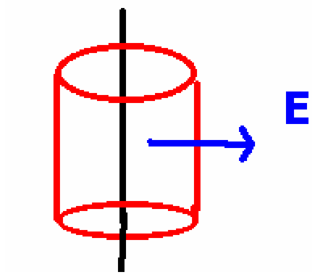
$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

the direction that E points is away from the charge (if it is positive).

Ok, now let's calculate this for an infinitely long line of charge.

Suppose we have a charge per unit length λ.

Choose a cylinder of height h and radius r centered on the wire.
No electric field passes through the ends



So the net flux through the cylinder is given by:

$$\Phi_E = E(\text{Area}) = E(2\pi rh)$$

The net charge enclosed here is given by:

$$Q = \lambda h$$

So we can then apply Gauss's Law:

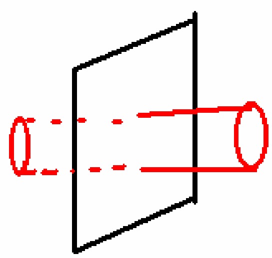
$$E(2\pi rh) = \frac{\lambda h}{\varepsilon_0}$$

And you solve this for E to obtain:

$$E = \frac{\lambda}{2\pi r\varepsilon_0}$$

And the electric field points away from the wire if the wire has a positive charge.

An infinite plane has a surface charge density σ (charge per unit area).
Choose a cylinder of area A' and orient it as I have shown.



No electric field passes through the sides: it all goes through the ends. On the positive side of the plane, the electric field is in the +x direction and on the negative side, it is in the –x direction. The total electric flux is then given by:

$$\Phi_E = 2EA'$$

The net charge enclose is given by:

$$Q_{enc} = \sigma A'$$

We can now write Gauss's law:

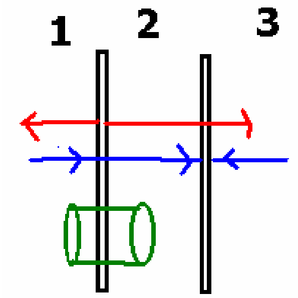
$$2EA' = \frac{\sigma A'}{\varepsilon_0}$$

Solve this for the electric field and you find:

$$E = \frac{\sigma}{2\varepsilon_0}$$

And now for a very important problem:
Two infinite planes are separated by a distance d.

The plane at x=0 has a surface charge density +σ while the plane located at x=d has a surface charge density -σ. Calculate the electric field inside and outside the planes. Choose the geometry as I have shown below:



In region 1 and 3, with the results of the previous problem, the net electric field is zero. In region 2, there is an electric field.

For the cylinder of area A' shown, the net electric flux is then given by:

$$\Phi_E = EA'$$

The electric charge enclosed is given by:

$$Q_{enc} = \sigma A'$$

We can now write Gauss's law:

$$EA' = \frac{\sigma A'}{\varepsilon_0}$$

Solve this for the electric field and you find:

$$E = \frac{\sigma}{\varepsilon_0}$$

The structure we have worked this for is the parallel plate capacitor and this shows that the electric field is zero outside the capacitor and it is uniform inside (so long as it is a parallel plate capacitor). This is fundamental to the rest of our work.

Electric Potential

While the electric field is extremely useful to find the force on a given charge, it is also desirable to know something about the energy landscape that a charge produces. We thus define the electric potential as the work per unit positive test charge to move a charge from infinity to a point close to a charge. The SI units for potential are Volts: 1 Volt is 1 Joule of energy per 1 Coulomb of charge (1 volt is a Joule per Coulomb). Mathematically, for discrete charges, the potential can be calculated as:

$$V(\vec{r}_p) = \sum_{i=1}^{N} k \frac{q_i}{|\vec{r}_{ip}|}$$

In general, calculations of the electric potential are somewhat simpler than calculating the electric field because the electric potential is a scalar, not a vector. Thus, it only has magnitude and not direction. As a very simple example, suppose a charge (+q) is located at x=+1 and a second charge (-q) is located at x=-1. Find the electric potential along the symmetry axis.

In this case, it is particularly easy to see that the electric potential is exactly zero along the symmetry axis. However, suppose you instead wished to calculate the electric potential along the +x axis at points beyond x=+1. In this case, the potential would be given by:

$$V(\vec{r}_p) = k\left[\frac{-q}{\sqrt{(x_p-(-1))^2}}\right] + k\left[\frac{+q}{\sqrt{(x_p-(+1))^2}}\right] = kq\left[\frac{1}{|x_p-1|} - \frac{1}{|x_p+1|}\right]$$

Now, if x_p is always positive and greater than 1, we have:

$$V(\vec{r}_p) = kq\left[\frac{1}{x_p-1} - \frac{1}{x_p+1}\right]$$

There is just a bit of mathematical magic here associated with the subtraction of these two fractions.

$$V(\vec{r}_p) = kq\left[\frac{1}{x_p-1} - \frac{1}{x_p+1}\right] = kq\left\{\frac{x_p+1-x_p+1}{x_p^2+1}\right\} = kq\frac{2}{1+x_p^2}$$

This would indicate that along the dipole axis, the electric potential drops off as 1/(distance^2).

Now normally we measure electric potential difference; in fact even electric potential is a potential difference except that one reference point is at infinity where the potential is assumed to be zero. There is a direct connection between electrostatic potential and electrostatic potential difference which is easily written for the case of uniform electric fields (as is the case inside a capacitor). This connection is:

$$|\vec{E}| = \left|-\frac{\Delta V}{\Delta x}\right|$$

Thus in regions of uniform electric field, it is particularly easy to obtain the electric field simply by measuring potential difference and dividing it by the distance between the two points of measurement. We'll do this in lab today with the electric dipole. In a more general situation, the electric field is, in fact related to the potential by a more complicated mathematical operation (the gradient operation). However, the essence of it is this: the electric field points in the direction of steepest descent on an energy landscape. The electric field points down a mountain in the quickest way possible. The electric potential surface is a contour line circling around the mountain.

When you sketch electric potential lines, think of them as rubber bands surrounding charges. The electric field always points at right angles to the rubber bands. The equipotential surfaces are parallel to conductors and thus electric field lines are perpendicular to a conducing surface.

I will show you how to draw equipotential surfaces for a few simply charge distributions in class.

Now it is electric potential difference that you normally measure with a voltmeter. This is the quantity that you have between the terminals of a battery and it is much easier to measure than the electric field. This is what we will measure today in the lab.

One final calculation involves the stored energy in a capacitor. Inside a parallel plate capacitor with a plate separation of d, the potential difference is given by:

$$\Delta V = Ed$$

We define the capacitance of the capacitor in terms of the efficiency of storing charge:

$$C = \frac{Q}{\Delta V}$$

For the parallel plate capacitor, our earlier work provides the potential difference in terms of charge density as:

$$\Delta V = Ed = \frac{\sigma}{\varepsilon_0} d$$

If the plates have an area A, then the total charge separation is given by:

$$Q = \sigma A$$

We can thus find the capacitance of the parallel plate capacitor as:

$$C = \frac{\sigma A}{\left(\frac{\sigma}{\varepsilon_0}\right) d} = \varepsilon_0 \frac{A}{d}$$

This means that the capacitance gets larger as the plate area increases and it also gets larger as the separation between the plates decreases. I have some large sheets of foil which we can test this with. The SI units of capacitance are Farads: 1 Farad (f) is 1 Coulomb of charge in the presence of a potential difference of 1 Volt. In general, with a material between the plates of a capacitor, the capacitance will increase if the material is not conducting.

Now we'll make a few finishing calculations for the capacitor. We wish to find the total work required to charge a capacitor. We can calculate this directly from the potential difference as follows:

(1) the work to bring the first charge across the plates is zero.

(2) the work to bring the second charge across the plates is $W_2 = qV_1 = q\left[\frac{q}{C}\right] = \frac{q^2}{C}$

(3) the work to bring the third charge across the plates is: $W_3 = qV_3 = q\left[\frac{2q}{C}\right] = 2\frac{q^2}{C}$

In general, you can approximate this for the Nth charge as:

$W_{N+1} = q\frac{Nq}{C} = N\frac{q^2}{C}$

The total work required is the sum of the individual works:

$$W = \sum_{i=1}^{N} i\frac{q^2}{C} = \frac{q^2}{C}\sum_{i=1}^{N} i = \frac{q^2}{C}\left[\frac{N^2}{2}\right]$$

(the sum is an arithmetic series).

Thus the stored energy inside the capacitor is given by:

$$U = \frac{Q^2}{2C}; Q = Nq$$

It is also easy from this to show that the energy density (energy per unit volume) would be given by:

$$U = \frac{1}{2}\varepsilon_0 E^2$$

which becomes very important to understand when dealing with electromagnetic waves.